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MATHEMATICS

Remarks Concerning Criteria for Uniqueness of Solutions of Ordinary Differential Equations

by

C. OLECH

Presented by T. WAŻEWSKI on May 16, 1960

1. Consider, for the ordinary differential equation, the initial problem

$$y' = f(x, y); \quad y(0) = 0, \tag{1.1}$$

where x, y are real variables and $f(x, y)$ is continuous on

$$\mathrm{D}: 0 \leqslant x \leqslant a, \ |y| \leqslant b \quad (a, b < +\infty).$$

Many criteria are known for uniqueness of (1.1). We mention here two of them ([3] p. 99 and 139).

Suppose $f(x, y)$ is continuous on D. The following inequality

$$|f(x, y_1) - f(x, y_2)| \leqslant h(x, |y_1 - y_2|) \tag{1.2}$$

on D represents a criterion for uniqueness of solution of (1.1), provided $h(x, u)$ satisfies one of the following two conditions.

Condition A (of Perron). $h(x, u)$ is continuous and non-negative for $0 \leqslant x \leqslant a$ and $0 \leqslant u$ and, for every $0 < \gamma < a$, $u(x) \equiv 0$ is the only differentiable on $0 \leqslant x < \gamma$ function which satisfies

$$u' = h(x, u) \quad \text{for} \quad 0 \leqslant x < \gamma \tag{1.3}$$

and the initial condition $u(0) = 0$.

Condition B (of Kamke). $h(x, u)$ is continuous and non-negative for $0 < x \leqslant a$ and $0 \leqslant u$ and, for every $0 < \gamma < a$, $u(x) \equiv 0$ is the only differentiable on $0 < x < \gamma$ and continuous on $0 \leqslant x < \gamma$ function which satisfies (1.3) on $0 < x < \gamma$ and for which $D_+ u(0) = \lim_{x \to 0+} u(x)/x$ exists and $D_+ u(0) = u(0) = 0$.

Evidently, the criterion of Kamke (Condition B) is more general than that of Perron (Condition A).

The main remark we want to make in this paper is contained in Theorem 1 and it may be sketched as follows: the criterion of Kamke for continuous function $f(x, y)$ is, in some sense, equivalent to that of Perron.

2. The criterion of Kamke is well illustrated by its special case, i.e. by the criterion of Nagumo

(2.1) $$|f(x, y_1) - f(x, y_2)| \leqslant |y_1 - y_2|/x.$$

Here $h(x, u) = u/x$, and this satisfies Condition B. The function u/x is near $(0, 0)$ unbounded and if $f(x, y)$ is continuous at $(0, 0)$, then the inequality (2.1) is automatically satisfied for some points from a neighbourhood of (0,0).

On the other hand, as the following example shows, the requirement on $f(x, y)$ to be continuous at (0,0) is, in some sense, necessary for the validity of criterion of Kamke or that of Nagumo. Indeed, put

(2.2) $$f(x, y) = 1 \text{ for } x < y,\ f(x, y) = y/x \text{ for } |x| > |y|,$$
$$f(x, y) = -1 \text{ for } -x > y\ (x \geqslant 0).$$

So defined right hand member of (1.1) is discontinuous at (0,0) but this satisfies the Carathéodory hypothesis ([2], p. 49) and also (2.1), though there is more than one solution of (1.1), provided we consider as a solution of (1.1) the function $y(x)$ continuous on $\langle 0, +\infty)$ and satisfying the Eq. (1.1) everywhere with the exception of $x = 0$.

3. Now we are going to prove the following

THEOREM 1. *Suppose* $f(x, y)$ *is continuous on* D *and suppose* (1.2) *holds with* $h(x, u)$, *satisfying Condition B.*

Then there exists another function $h_f(x, u)$ *satisfying Condition A and for which the following inequality holds*

(3.1) $$|f(x, y_1) - f(x, y_2)| \leqslant h_f(x, |y_1 - y_2|) \quad on \quad \mathrm{D}.$$

Proof. Put

(3.2) $$h_f(x, u) = \sup_{|y_1 - y_2| = u} |f(x, y_1) - f(x, y_2)| \text{ where } |y_1|, |y_2| \leqslant b$$

for $0 \leqslant x \leqslant a$ and $0 \leqslant u \leqslant 2b$, and

$$h_f(x, u) = h_f(x, 2b) \text{ for } 0 \leqslant x \leqslant a \text{ and } u > 2b.$$

Since $f(x, y)$ is continuous on D, $h_f(x, u)$ is continuous on $(0 \leqslant x \leqslant a;\ 0 \leqslant u)$.

Evidently, (3.1) holds for the function so defined. Further, by (1.2) and (3.2), we get the inequality

(3.3) $$h_f(x, u) \leqslant h(x, u) \text{ for } 0 < x \leqslant a, \quad 0 \leqslant u \leqslant 2b.$$

Now we consider the equation

(3.4) $$u' = h_f(x, u).$$

Since $h_f(x, y)$ is continuous, there exists the maximum solution $U(x)$ of (3.4) issuing from (0,0). We are going to prove that $U(x) \equiv 0$. Suppose that there exists $x_0 > 0$ such that $U(x_0) > 0$. Without loss of generality we may suppose that $U(x) < 2b$ for $0 \leqslant x \leqslant x_0$. Then, owing to the inequality following (3.3)

$$U'(x) \leqslant h(x, U(x)) \quad \text{for} \quad 0 < x \leqslant x_0$$

we have (see [1], p. 10 and [5])

$$U(x) \geqslant \varphi(x) \geqslant 0 \text{ for } x \leqslant x_0, \tag{3.5}$$

where by $\varphi(x)$ we denote the minimal solution of (1.3) passing through $(x_0, U(x_0))$. By (3.5) we get that $\varphi(x)$ may be continued on $0 \leqslant x \leqslant x_0$ and that $\varphi(0) = 0$. Further, since $h_f(x, u)$ is continuous at (0,0) and $h_f(00) = 0$, then $D_+ U(0)$ exists and $D_+ U(0) = 0$ and, by (3.5), also $D_+ \varphi(0)$ exists, and $D_+ \varphi(0) = 0$. But we assumed that $h(x, u)$ satisfies Condition B, therefore $\varphi(x) \equiv 0$. This contradicts the assumption of $\varphi(x_0) = U(x_0) > 0$. Hence, $U(x) \equiv 0$. Thus, we have shown that $h_f(x, u)$ satisfies Condition A and at the same time proved Theorem 1 completely.

4. In this section we give a modification of Theorem 1.

THEOREM 2. *Suppose $h(x, u)$ satisfies Condition B and suppose* (1.2) *holds on* D. *Let $g(x, u)$ be an arbitrary continuous function for $0 \leqslant x \leqslant a$ and $u \geqslant 0$ and such that $g(0,0) = 0$ and let the inequality*

$$|f(x, y_1) - f(x, y_2)| \leqslant g(x, |y_1 - y_2|) \tag{4.1}$$

hold on D.

Put

$$h^*(x, u) = \min\, (g(x, u),\ h(x, u)). \tag{4.2}$$

Then

$$|f(x, y_1) - f(x, y_2)| \leqslant h^*(x, |y_1 - y_2|)$$

on D *and $h^*(x, u)$ satisfies the slightly modified Condition A, that is $h^*(x, u)$ continuous for $0 < x \leqslant a$ and $0 \leqslant u$ and, for every $0 < \gamma < a$, $u(x) \equiv 0$ is the only continuous on $0 \leqslant x \leqslant \gamma$ and differentiable on $0 < x \leqslant \gamma$ function which satisfies*

$$u' = h^*(x, u) \text{ for } 0 < x \leqslant \gamma \tag{4.3}$$

and the initial condition $u(0) = 0$.

The proof of this theorem is quite similar to that of Theorem 1. We observe only that since $h^*(x, u)$ is bounded on each bounded set then to any $u_0 \geqslant 0$ there exists function $u(x)$ continuous on $\langle 0, \gamma)$ for some $\gamma > 0$ and differentiable on $(0, \gamma)$ which satisfies (4.3) for $0 < x < \gamma$ and such that $u(0) = u_0$. Hence, there exists also the maximum solution $U(x)$ of (4.3) issuing from (0,0) and since $\lim h^*(x, u)$ exists and equals 0 there exist $D_+ U(0)$ and $D_+ U(0) = 0$. Further, owing to the inequality $h^*(x, u) \leqslant h(x, u)$, $U(x) \equiv 0$.

Remark 1. In the proofs of uniqueness criteria and also in other reasoning in which assumption (1.2) appears (see [6]) we use (1.2) not in the whole set D but only in some subset of D, say

$$\text{C:}\quad 0 \leqslant x \leqslant a, \qquad |y| \leqslant Mx.$$

Thus, for some purposes it suffices to consider the continuous function $g(x)$ for $0 \leqslant x \leqslant a$, such that $g(0) = 0$ and

$$g(x) \geqslant \sup_{|y_1|,\,|y_2| \leqslant Mx} |f(x, y_1) - f(x, y_2)|$$

and the function

$$\bar{h}(x, u) = \min (g(x),\ h(x, u))$$

instead of $g(x, u)$ and $h^*(x, u)$, respectively.

For so defined function $\bar{h}(x, u)$ the inequality

$$|f(x, y_1) - f(x, y_2)| \leqslant \bar{h}(x, |y_1 - y_2|)$$

does not hold everywhere on D, but only on a subset of D in which it is needed, i.e. on C (see, e.g. [6]). The equation $u' = \bar{h}(x, u)$ has unique solution $u(x) \equiv 0$ issuing from (0,0).

5. Now let us consider two approximate solutions of (1.1), that is the continuous functions $q_i(x)$ $(i = 1, 2)$ defined on $0 \leqslant x \leqslant \alpha \leqslant a$ and such that $|q_i(x)| \leqslant \beta < b$ on $0 \leqslant x \leqslant \alpha$ and

(5.1) $\quad |D_+ q_i(x) - f(x, q_i(x))| \leqslant \delta \quad \text{and} \quad |q_i(0)| < \delta \quad (0 < x \leqslant \alpha;\ i = 1, 2)$

We shall now prove the following

THEOREM 3. *Suppose $f(x, y)$ is continuous on D and suppose the inequality* (1.2) *holds, where the function $h(x, u)$ satisfies Condition B.*

Then to any $\varepsilon > 0$ there exists $\delta > 0$ such that for every pair of approximate solutions of problem (1.1) *satisfying* (5.1) *we have the inequality*

$$|q_1(x) - q_2(x)| \leqslant \varepsilon \quad \textit{for} \quad 0 \leqslant x \leqslant \alpha. \tag{5.2}$$

Proof. First, let us observe that owing to (5.1) the functions $q_i(x)$ satisfy the Lipschitz condition and therefore $|q_i(x)|$ as well as $|q_1(x) - q_2(x)|$ are absolutely continuous, and also there exists a constant M such that

$$|q_i(x)| \leqslant \delta + Mx \quad (i = 1, 2;\ 0 \leqslant x \leqslant \alpha). \tag{5.3}$$

Let us now define the functions $g_\delta(x)$ as follows:

$$g_\delta(x) = \sup_{|y_1|,\ |y_2| \leqslant \delta + Mx} |f(x, y_1) - f(x, y_2)| \tag{5.4}$$

and let us put

$$h_\delta(x, u) = \min (g_\delta(x), h(x, u)).$$

On the basis of (5.3), (5.4) and (1.2) we have

$$|f(x, q_1(x)) - f(x, q_2(x))| \leqslant \min (g_\delta(x), h(x, |q_1(x) - q_2(x)|)) \tag{5.5}$$

By (5.1) and by the inequality

$$\bar{D}_+ |q_1(x) - q_2(x)| \leqslant |D_+ q_1(x) - D_+ q_2(x)|$$

we obtain

$$\bar{D}_+ |q_1(x) - q_2(x)| \leqslant |f(x, q_1(x)) - f(x, q_2(x))| + 2\delta$$

and, using (5.5),

$$\bar{D}_+ |q_1(x) - q_2(x)| \leqslant h_\delta(x, |q_1(x) - q_2(x)|) + 2\delta. \tag{5.6}$$

Now let us denote by $U_\delta(x)$ the maximum solution of

$$u' = h_\delta(x, u) + 2\delta$$

issuing from the point $(0,2\delta)$. For sufficiently small δ these solutions exist at least on $0 \leqslant x \leqslant a$. From (5.6) and (5.1), [1], we have

$$|q_1(x) - q_2(x)| \leqslant U_\delta(x) \text{ for } 0 \leqslant x \leqslant a. \tag{5.7}$$

But, if $\delta \to 0$, then $U_\delta(x)$ tends uniformly over $\langle 0, a \rangle$ to the maximum solution $U(x)$ of the equation

$$u' = h_0(x, u) = \min(g_0(x), h(x, u)),$$

issuing from (0,0).

Since $U(0) = 0$ then, as follows from Remark 1, $U(x) \equiv 0$. Therefore there exists $\delta > 0$ such that

$$U_\delta(x) < \varepsilon \text{ for } 0 \leqslant x \leqslant a.$$

This, together with (5.7), proves Theorem 3 completely.

Remark 2. Note that the equation

$$u' = h(x, u) + \varepsilon,$$

where $\varepsilon > 0$, and $h(x, u)$ satisfies only Condition B (for example if $h(x, u) = u/x$) may not admit any solution issuing from (0,0). Thus, it seems that the idea of replacing the function $h(x, u)$ by more regular function is the main point of the proof of Theorem 3.

6. In this section we shall discuss the case when $h(x, u)$ is not continuous but satisfying the Carathéodory hypothesis, that is, we consider the following condition ([2], p. 49).

Condition C (of Coddington and Levinson). $h(x, u)$ is continuous in u for each fixed x and Lebesgue measurable in x for each fixed u; for each bounded subset S of the set $0 \leqslant x \leqslant a$, $u \leqslant 0$ there exists Lebesgue measurable function $\chi_S(x)$ for $0 \leqslant x \leqslant a$ and Lebesgue integrable on each interval $\gamma \leqslant x \leqslant a$, $\gamma > 0$ and such that $h(x, u) \leqslant \chi_S(x)$ on S, $u(x) \equiv 0$ is the only absolutely continuous function satisfying (1.3) almost everywhere on $0 \leqslant x \leqslant \gamma$ for every $\gamma > 0$, and for which $D_+ u(0)$ exists and $D_+ u(0) = u(0) = 0$.

It is well known that if $f(x, y)$ is continuous then (1.2) together with Condition C, guarantees uniqueness of (1.1).

The following generalization of Theorem 1 is valid.

THEOREM 4. *Suppose $f(x, y)$ is continuous on* D *and satisfies* (1.2), *where $h(x, u)$ fulfills Condition* C.

Then there exists another function $h_f(x, u)$ (the same as in Theorem 1) *satisfying Condition* A *for which* (3.1) *holds.*

Similarly, Theorems 2 and 3 may be generalized for the case, when $h(x, u)$ is supposed to fulfil Condition C instead of B.

Remark 3. Notice that if $h(x, u)$ is nondecreasing in u, then we may put, in the proof of Theorem 1 as well as of Theorem 4, $h_f(x, u) = \overline{g}(x, u)$ for $0 \leqslant x \leqslant a$,

$0 \leqslant u \leqslant 2b$ and $h_f(x,u) = h_f(x,2b)$ for $u > 2b$, where $g(x,u) = \sup_{|y_1 - y_2| \leqslant u} |f(x,y_1) - f(x,y_2)|$.

Then $h_f(x,u)$ is also nondecreasing in u. Hence, in the proofs of the convergence of successive approximations of Picard's method for (1.1) when (1.2) is supposed with $h(x,u)$ nondecreasing in u and satisfying Condition C ([2] p. 54 and also [4]) one can replace $h(x,u)$ by $h_f(x,u)$ also nondecreasing in u but satisfying more restrictive Condition A (see [7]).

7. For the sake of simplicity we have supposed throughout the paper that y is real but our results remain valid if y belongs to n-dimensional Euclidean space and also, with some changes, if y belongs to the complete linear Banach space.

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MATHEMATICS

On the Existence and Uniqueness of Solutions of an Ordinary Differential Equation in the Case of Banach Space

by

C. OLECH

Presented by T. WAŻEWSKI on May 16, 1960

1. Let B be a linear complete Banach space with homogeneous norm. We denote by $|z|$ the norm of z, if $z \in B$. Consider the differential equation

$$y' = f(x, y), \quad (' = d/dx), \tag{1.1}$$

where x is real, $y \in B$ and $f(x, y)$ is a continuous function with values belonging to B.

We give in this paper a sufficient condition for the existence and uniqueness of solutions of (1.1). This condition has the same form as that given by E. A. Coddington and N. Levinson ([2], p. 49) for the uniqueness of (1.1) for y real.

Our results represent the generalization of those of T. Ważewski [5]. We are able to prove Ważewski's result in more general form because of an idea of replacing the function $h(x, u)$ satisfying (i) and (ii) (of sec. 2) by another one more regular and having the same properties (see Lemmas 1 and 2).

2. Consider the equation

$$u' = h(x, u), \tag{2.1}$$

where x, u are real variables and $h(x, u)$ is a real non-negative function defined on

$$V\colon \quad 0 \leqslant u, \quad 0 \leqslant x \leqslant a \quad (a > 0).$$

Let us assume the following conditions for $h(x, u)$:

(i) $h(x, u)$ is continuous in u for every fixed x and Lebesgue measurable in x for every fixed u; for every bounded subset S of V there exists a Lebesgue-measurable function $\delta_s(x)$ defined on $0 \leqslant x < a$, such that $h(x, u) \leqslant \delta_s(x)$ for $(x, u) \in S$ and $\delta_s(x)$ is Lebesgue integrable on $\gamma \leqslant x \leqslant a$ for every $\gamma > 0$.

(ii) For each γ, $0 < \gamma < a$, the identically zero function is the only absolutely continuous function on $0 \leqslant x \leqslant \gamma$ which satisfies (2.1) almost everywhere on $0 \leqslant x \leqslant \gamma$, and such that $D_+ u(0)$ exists and

$$u(0) = D_+ u(0) = 0^{*)}.$$

*) By $D_+ u(0)$ we denote the right hand derivative of $u(x)$ at $x = 0$.

3. The main thesis of this paper is the following:

THEOREM 1. *Consider Eq.* (1.1) *and suppose* $f(x, y)$, *where* $y, f(x, y) \in B$, *is continuous and bounded on*

$$W: \quad 0 \leqslant x - x_0 \leqslant a, \quad |y - y_0| \leqslant b \quad (0 < a, b < +\infty)$$

and let $M < +\infty$ *be the upper bound of* $|f(x, y)|$ *on* W.

Suppose $h(x, u)$ *satisfies* (i) *and* (ii), *and*

$$|f(x, y_1) - f(x, y_2)| \leqslant h(x - x_0, |y_1 - y_2|) \text{ on } W. \tag{3.1}$$

Then there exists a solution of (1.1) *issuing from* (x_0, y_0) *and determined on* $x_0 \leqslant x \leqslant x_0 + \alpha$, *where* $\alpha = \min(a, b/M)$ *and is unique.*

Proof (the first part). Without loss of generality we may suppose that $x_0 = 0$ and $y_0 = 0$. Following T. Ważewski [5], we construct, applying an idea of Carathéodory, the sequence $q_n(x)$ convergent to the solution of (1.1)

Adopting the denotations of [5] let us put $c_n = \alpha/n$ and $J(k, n) = \langle kc_n, (k+1)c_n \rangle$ $(k = 0, 1, \ldots, n-1;\ n = 1, 2, \ldots)$. We define $q_n(x)$ inductively:

$$q_n(0) = 0 \quad (n = 1, 2, \ldots), \tag{3.2}$$

$$q_n(x) = q_n(kc_n) + \int_{kc_n}^{x} f(x, q_n(kc_n))\, dx \text{ for } x \in J(k, n), \tag{3.3}$$

$$(k = 0, 1, \ldots, n-1; \quad n = 1, 2, \ldots).$$

Thus we have

$$q_n(x) = \int_0^x f(x, \lambda_n(x))\, dx \text{ for } 0 \leqslant x \leqslant \alpha \text{ and } n = 1, 2, \ldots, \tag{3.3'}$$

where

$$\lambda_n(x) = q_n(kc_n) \text{ for } x \in J(k, n);\ k = 0, 1, \ldots, n-1;\ n = 1, 2, \ldots$$

It may be easily proved that $q_n(x)$ are defined on $0 \leqslant x \leqslant \alpha$ and that

$$|q_n(x)| \leqslant Mx \text{ for } 0 \leqslant x \leqslant \alpha \text{ and } n = 1, 2, \ldots, \tag{3.4}$$

where M is the upper bound of $|f(x, y)|$ on W. Similarly, by (3.3), we get

$$|q_n(x) - q_n(kc_n)| \leqslant M(x - kc_n) \leqslant Mc_n \text{ for } x \in J(k, n), \tag{3.5}$$
$$k = 0, 1, \ldots, n-1; \quad n = 1, 2, \ldots$$

It follows from (3.3), (3.5) and from the assumption (3.1) that

$$|D_+ q_n(x) - f(x, q_n(x))| \leqslant h(x, |q_n(x) - q_n(kc_n)|) \text{ for } x \in J(k, n), \tag{3.6}$$
$$k = 0, 1, \ldots, n-1;\ n = 1, 2, \ldots$$

Now let us put

$$\varepsilon_n(x) = \sup_{0 \leqslant u \leqslant Mc_n} h(x, u) \quad (0 \leqslant x \leqslant \alpha). \tag{3.7}$$

Evidently, owing to (i), $\varepsilon_n(x)$ $(n = 1, 2, \dots)$ is Lebesgue measurable function, and Lebesgue integrable on $\gamma \leqslant x \leqslant \alpha$ for each $0 < \gamma < \alpha$,

(3.8) $$\varepsilon_{n+1}(x) \leqslant \varepsilon_n(x) \quad (0 < x \leqslant \alpha; \quad n = 1, 2, \dots),$$

and

(3.9) $$\lim_{n \to \infty} \varepsilon_n(x) = 0 \quad \text{almost everywhere on} \quad 0 < x \leqslant \alpha.$$

By (3.5), (3.6) and (3.7) we get that

(3.10) $$|D_+ q_n(x) - f(x, q_n(x))| \leqslant \varepsilon_n(x) \quad \text{for} \quad 0 < x \leqslant \alpha.$$

Now let us estimate the difference $D_+ q_n(x) - D_+ q_m(x)$, for $m > n$. By (3.10) we get

$$|D_+ q_n(x) - D_+ q_m(x)| \leqslant \varepsilon_n(x) + \varepsilon_m(x) + |f(x, q_n(x)) - f(x, q_m(x))|.$$

Hence, owing to (3.1) and (3.8), the following inequality holds

(3.11) $$|D_+ q_n(x) - D_+ q_m(x)| \leqslant h(x, |q_n(x) - q_m(x)|) + 2\varepsilon_n(x).$$

If we denote by

(3.12) $$\mu(x) = \sup_{|y_1|, |y_2| \leqslant Mx} |f(x, y_1) - f(x, y_2)|,$$

we get from (3.4) and (3.3)

(3.13) $$|D_+ q_n(x) - D_+ q_m(x)| \leqslant \mu(x) \quad \text{for} \quad 0 \leqslant x \leqslant \alpha; \quad n, m = 1, 2, \dots$$

The function $\mu(x)$ is bounded. We shall prove that $\mu(x)$ is lower semicontinuous for $0 < x \leqslant \alpha$ and continuous for $x = 0$. Indeed, let us take $\varepsilon > 0$ and fix $0 < x_0 \leqslant \alpha$. Owing to (3.12) there exist y_1 and y_2 such that

(3.14) $$\mu(x_0) - \varepsilon/2 \leqslant |f(x_0, y_1) - f(x_0, y_2)|.$$

Since $f(x, y)$ is continuous, there exists a positive number δ such that

$$|f(x_0, y_1) - f(x, y_1^*)| < \varepsilon/4 \quad \text{and} \quad |f(x_0, y_2) - f(x, y_2^*)| < \varepsilon/4$$

if $x_0 - \delta < x < x_0 + \delta$ and $|y_1 - y_1^*| < \delta$, $|y_2 - y_2^*| < \delta$.

The last and (3.14) imply that for each $x_0 - \delta < x < x_0 + \delta$ there exist $\bar{y}_1$ and $\bar{y}_2$, $|\bar{y}_1| < Mx$ and $|\bar{y}_2| < Mx$ and such that

$$\mu(x_0) - \varepsilon < |f(x, \bar{y}_1) - f(x, \bar{y}_2)| \leqslant \mu(x).$$

Hence, $\mu(x)$ is lower semicontinuous.

Because of the continuity of $f(x, y)$ to any $\varepsilon > 0$ there exists $\delta > 0$ such that $|f(x, y_1) - f(x, y_2)| < \varepsilon$ for $0 \leqslant x \leqslant \delta$ and $|y_1| < Mx$ and $|y_2| < Mx$. Hence, $\mu(x) < \varepsilon$ for $0 \leqslant x \leqslant \delta$. This means that $\mu(x)$ is continuous for $x = 0$.

Since $\mu(x)$ is semicontinuous and bounded, there exists the integral

$$m(x) = \int_0^x \mu(x)\, dx$$

and since $\mu(x)$ is continuous for $x=0$, there exists the limit

(3.15) $$\lim_{x\to 0+} \frac{m(x)}{x} = \lim_{x\to 0+} \frac{1}{x} \int_0^x \mu(x)\,dx = \mu(0) = 0.$$

4. In this section we prove two lemmas which we will use in order to achieve the proof of Theorem 1 [4].

LEMMA 1. *Suppose* $h(x,u)$ *satisfies* (i) *and* (ii) *on* V.

Let $s(x)$ *be absolutely continuous on* $0 \leqslant x \leqslant a$, $D_+ s(0) = 0 = s(0)$ *and* $\sigma(x) = s'(x) \geqslant 0$. *Put*

$$g(x,u) = \min(\sigma(x),\ h(x,u)).$$

Then $u(x) \equiv 0$ *is the only absolutely continuous function satisfying almost everywhere the equation*

(4.1) $$u' = g(x,u)$$

and the initial condition $u(0) = 0$,

The proof of Lemma 1 may be deduced from the consideration of paper [4] (Theorem 1, Remark 1, sec. 7).

LEMMA 2. *Suppose* $s(x)$ *and* $\sigma(x)$ *satisfy the same assumptions as in Lemma* 1. *Let* $\eta_n(x)$ *be the non-increasing sequence of positive Lebesgue measurable functions on* $0 \leqslant x \leqslant a$ *and let* $\eta_n(x)$ *tend to zero almost everywhere on* $0 \leqslant x \leqslant a$.

Put

$$g_n(x,u) = \min(\sigma(x),\ h(x,u) + \eta_n(x)) \quad (n = 1, \ldots).$$

Denote by $u_n(x)$ $(n = 1, 2, \ldots)$ *the maximum solution of*

(4.2) $$u' = g_n(x,u)\ ^*)$$

issuing from (0,0).

Then $u_n(x) \to 0$ *uniformly on* $0 \leqslant x \leqslant a$ *if* $n \to +\infty$.

Proof. The sequence $u_n(x)$ is non-increasing and equicontinuous and therefore it tends uniformly over $\langle 0, a\rangle$ to a function $u(x)$. Evidently, $u(x)$ is a solution of (4.1) and $u(0) = 0$. Thus, from Lemma 1 we get $u(x) \equiv 0$. This completes the proof of Lemma 2.

5. Proof of Theorem 1 (second part). Let us put, in Lemma 2,

$$s(x) = m(x), \qquad \sigma(x) = \mu(x), \qquad \eta_n(x) = 2\,\varepsilon_n(x)$$

and

(5.1) $$g_n(x,u) = \min(\mu(x),\ h(x,u) + 2\,\varepsilon_n(x)),$$

where $\mu(x)$ is defined by (3.12), $\varepsilon_n(x)$ — by (3.7). It follows from sec. **3** that these functions satisfy the assumptions of Lemma 2.

By (3.11), (3.13) and (5.1) we have

(5.2) $$D_+ |q_n(x) - q_m(x)| \leqslant |D_+ q_n(x) - D_+ q_m(x)| \leqslant g_n(x, |q_n(x) - q_m(x)|).$$

*) Since $g_n(x,u)$ satisfies the Carathéodory hypothesis on V therefore the maximum solution of (4.2) exists, and because $g_n(x,u) \leqslant \sigma(x)$, $u_n(x)$ may be continued on $0 \leqslant x \leqslant a$ and $u_n(x) \leqslant s(x)$.

The last inequality implies [1], [3] the following one

$$|q_n(x) - q_m(x)| \leqslant u_n(x) \quad (n < m). \tag{5.3}$$

This, together with Lemma 2, shows that $q_n(x)$ is uniformly convergent. Suppose $q(x)$ is the limit of $q_n(x)$. We prove that $q(x)$ is a solution of (1.1) issuing from (0,0). Since $f(x, y)$ is continuous, the sequence $f(x, q_n(x))$ is convergent, too, and to the function $p(x) = f(x, q(x))$. Let us estimate the difference

$$|\Delta q(x) - \int_x^{x+r} p(x)\,dx| = |q(x+r) - q(x) - \int_x^{x+r} f(x, q(x))\,dx|.$$

We have

$$|\Delta q(x) - \int_x^{x+r} p(x)\,dx| \leqslant |\Delta q(x) - \Delta q_n(x)| + |\Delta q_n(x) - \int_x^{x+r} f(x, q(x))\,dx| + \\ + \int_x^{x+r} |f(x, q_n(x)) - f(x, q(x))|\,dx.$$

By (5.3) we get $|\Delta q(x) - \Delta q_n(x)| \leqslant |q(x+r) - q_n(x+r)| + |q(x) - q_n(x)| \leqslant \leqslant u_n(x+r) + u_n(x)$, by (3.10), (3.3), (3.4) and (3.12) $|\Delta q_n(x) - \int_x^{x+r} f(x, q_n(x))\,dx| \leqslant \leqslant \int_x^{x+r} \min(\mu(x), \varepsilon_n(x))\,dx$ and at last by (3.1), (3.4) and (3.12) $\int_x^{x+r} |f(x, q_n(x)) - - f(x, q(x))|\,dx \leqslant \int_x^{x+r} \min(\mu(x), h(x, |q_n(x) - q(x)|))\,dx$.

Hence, we have

$$|\Delta q(x) - \int_x^{x+r} p(x)\,dx| \leqslant u_n(x+r) + u_n(x) + \int_x^{x+r} \min(\mu(x), \varepsilon_n(x))\,dx + \\ + \int_x^{x+r} \min(\mu(x), h(x, |q(x) - q_n(x)|)\,dx.$$

Since the right hand terms of the above inequality tend to zero as $n \to \infty$, we have

$$\Delta q(x) = \int_x^{x+r} p(x)\,dx = \int_x^{x+r} f(x, q(x))\,dx \quad \text{for} \quad 0 \leqslant x \leqslant a.$$

Since r may be arbitrary and because $p(x)$ is continuous the above equality implies

$$q'(x) = f(x, q(x)) \quad \text{for} \quad 0 \leqslant x \leqslant a.$$

Thus, $q(x)$ is a solution of (1,1) on $0 \leqslant x \leqslant a$. The uniqueness may be proved by the usual arguments and we omit here the details. Therefore Theorem 1 is proved.

6. As in [5] the following theorem of the local character may be proved.

THEOREM 2. *Suppose $f(x, y)$, where $y, f(x, y) \in B$, is continuous in a neighbourhood of (x_0, y_0) (we do not suppose the boundedness of $f(x, y)$) and suppose* (3.1)), *where $h(x, u)$ satisfies* (i) *and* (ii).

Then there exists a positive number δ such that there exists a unique solution $q(x)$ of (1.1) *issuing from (x_0, y_0) and determined on $x_0 \leqslant x \leqslant x_0 + \delta$.*

7. In this section we give an analogous theorem to Theorem 3 of paper [5].

THEOREM 3. *Suppose $f(x, y)$, where $y, f(x, y) \in B$, is continuous and not necessarily bounded on* W. *Let M be a constance such that for some $\delta > 0$*

$$|f(x, y)| \leqslant M \quad \textit{for} \quad x_0 \leqslant x \leqslant x_0+\delta \qquad \textit{and } |y| \leqslant M\delta.$$

Put

$$\begin{cases} g(x, u) = \min\left(M, h(x-x_0, u)+|f(x, y_0)|\right) \ \textit{if } x_0 \leqslant x \leqslant x_0+\delta \\ g(x, u) = h(x-x_0, u)+|f(x, y_0)| \ \textit{if } x_0+\delta < x \leqslant x_0+a, \end{cases} \tag{7.1}$$

where $h(x, u)$ satisfies (i) *and* (ii), *and $f(x, y)$ fulfills* (3.1) *on* W.

Denote by $U(x)$ the maximum solution of

$$u' = g(x, u), \qquad u(x_0) = 0^{*)}$$

and suppose

$$U(x) < \delta \quad \text{for} \quad x_0 \leqslant x < c+x_0 \leqslant x_0+a.$$

Then there exists a solution $q(x)$ of (1.1) *issuing from (x_0, y_0) and defined on $x_0 \leqslant x < x_0+c$.*

Proof. In virtue of Theorem 2 there exists a solution $q(x)$ of (1.1) issuing from (x_0, y_0) and defined on $x_0 \leqslant x < x_0+\eta$. Suppose $\eta < c$. We have the following inequalities:

$$D_+|q(x)-y_0| \leqslant |q'(x)| \leqslant |f(x, q(x))-f(x, y_0)|+|f(x, y_0)| \leqslant h(x-x_0, \\ |q(x)-y_0|)+|f(x, y_0)|$$

and

$$D_+|q(x)-y_0| \leqslant |f(y, q(x))| \leqslant M \quad \text{for} \quad x \leqslant x \leqslant x_0+\min(\eta, \delta).$$

Therefore, owing to (7.1), we have

$$D_+|q(x)-y_0| \leqslant g(x, |q(x)-y_0|).$$

It follows from the last inequality that

$$|q(x)-y_0| \leqslant U(x) \quad \text{for} \quad x_0 \leqslant x < x_0+\eta. \tag{7.2}$$

But from the inequality $|q'(x)| \leqslant g(x, |q(x)-y_0|)$ we conclude that $|q'(x)|$ is bounded by a Lebesgue integrable function, thus there exists the limit $\lim\limits_{x\to x_0+\eta-0} q(x) = y$, and, by (7.2), $(x_0+\eta, y)$ belongs to the interior of W. Hence, $q(x)$ may be continued on the whole interval $x_0 \leqslant x < x_0+c$, at least. Because of the assumption (3.1) this solution is unique. Thus Theorem 3 is proved.

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*) It may be easily verified that $g(x, u)$ satisfies the Carathéodory hypothesis and therefore the solution $U(x)$ exists.

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MATHEMATICS

A Note on Singular Integrals of Vector-valued Functions

by

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Let $\mathfrak{X}$ be a Banach space with a norm $\|\cdot\|$. Let further $x(s)$ denote a vector-valued function from an interval $\langle a, b\rangle$ to $\mathfrak{X}$, and $(B)\int_b^a x(s)\,ds$ — the Bochner integral of the function $x(s)$ over $\langle a, b\rangle$. $f(s)$, $g(s)$ and $K(s,t)$ always denote real functions, their measurability and integrability are always understood in Lebesgue sense, $\chi_S(s)$ — the characteristic function of the set S.

$M(u)$ and $N(u)$ are complementary N-functions, as well as $L^M\langle a, b\rangle$ is the corresponding Orlicz space with norm $\|\cdot\|_M$. The set $\{x(\cdot): x(s)$ is strongly measurable in $\langle a, b\rangle\}$,

$$\|x(\cdot)\|_M = \sup_{\varrho(g,N)\leqslant 1} \int_a^b \|x(s)\,g(s)\|\,ds < \infty,$$

where $\varrho(g,N)=\int_a^b N[g(s)]\,ds$, is an Orlicz space constitued by vector-valued functions.

$B_{2\pi}^M(\mathfrak{X})$ — the space of functions belonging to $B_{2\pi}^M(\langle 0, 2\pi\rangle; \mathfrak{X})$ which are periodically extended to the whole straight line. For $x(\cdot)\in B_{2\pi}^M(\mathfrak{X})$, let $S_n(x,s)$ denote the n-th partial sum of the Fourier series of $x(s)$, and $\omega_M(x,\delta)=\sup_{|h|\leqslant\delta}\|x(\cdot+h)-x(\cdot)\|_M$.

In general, we use the fundamental notions of functional analysis as in [1], and of the theory of vector-valued functions and Orlicz spaces as in [3] and [4].

1.0. We shall now proceed to generalize some lemmas of Orlicz on the convergence in mean of singular integrals in $L^M\langle a, b\rangle$ ([6], pp. 129—132) to the case of vector-valued functions omitting the condition Δ_2.

1.1. LEMMA. *Let $K(s,t)$ be a measurable function in $\langle a, b\rangle^2$ such that $\int_a^b |K(s,t)|\,ds \leqslant A$ for a.e. $t\in\langle a, b\rangle$, $\int_a^b |K(s,t)\,dt \leqslant A$ for a.e. $s\in\langle a, b\rangle$, where A is a constant independent of s and t. Then for every $x(\cdot)\in B^M(\langle a, b\rangle; \mathfrak{X})$, the integral $\sigma(x,t)=(B)\int_a^b x(s)\,K(s,t)\,ds$ exists for a.e. $t\in\langle a, b\rangle$, and $\|\sigma(x,\cdot)\|_M \leqslant 2A\,\|x(\cdot)\|_M$, $\sigma(x,\cdot)\in B^M(\langle a, b\rangle; \mathfrak{X})$.*

Proof. The function $\sigma(x, t)$ exists for a.e. $t \in \langle a, b\rangle$ and is strongly measurable in $\langle a, b\rangle$ ([3], p. 48, Th. 3.6.7.). Let us assume that $x(s) \neq 0$ and $\int_a^b |K(s, t)|\, ds \neq 0$ a.e. in $\langle a, b\rangle$; in the other case the Lemma is obvious.

$$\text{If } x_1(s) = \frac{x(s)}{\|x(\cdot)\|_M} \quad \text{and} \quad K_1(s, t) = \frac{K(s, t)}{A}, \quad \text{then}$$

$$\int_a^b M(\|x_1(s)\|)\, ds \leqslant \|x_1(\cdot)\|_M = 1, \qquad \int_a^b |K_1(s, t)|\, ds \leqslant 1 \quad \text{and} \quad \int_a^b |K_1(s, t)|\, dt \leqslant 1.$$

From the convexity of $M(u)$ and the integral Jensen inequality, we have

$$M\left[\frac{1}{A}\|\sigma(x_1, t)\|\right] \leqslant M\left[\int_a^b \|x_1(s)\|\, |K_1(s, t)|\, ds\right] \leqslant$$

$$\leqslant \left(\int_a^b |K_1(s, t)|\, ds\right) M\left[\frac{\int_a^b \|x_1(s)\|\, |K_1(s, t)|\, ds}{\int_a^b |K_1(s, t)|\, ds}\right] \leqslant \int_a^b M(\|x_1(s)\|)\, |K_1(s, t)|\, ds,$$

therefore

$$\int_a^b M\left[\frac{1}{A}\|\sigma(x_1, t)\|\right] dt \leqslant \int_a^b M(\|x_1(s)\|)\, ds \leqslant 1.$$

Hence, for $\varrho(g, N) \leqslant 1$, we have

$$\int_a^b \|\sigma(x, t)\, g(t)\|\, dt \leqslant A\, \|x(\cdot)\|_M \left\{\int_a^b M\left[\frac{1}{A}\|\sigma(x_1, t)\|\right] dt + \right.$$

$$\left. + \int_a^b N[g(t)]\, dt\right\} \leqslant 2A\, \|x(\cdot)\|,$$

or

$$\|\sigma(x, \cdot)\|_M \leqslant 2A\, \|x(\cdot)\| \quad \text{and} \quad \sigma(x, \cdot) \in B_M(\langle a, b\rangle; \mathcal{X}).$$

1.2. Lemma. *Let $K_n(s, t)$ $(n = 1, 2, \ldots)$ satisfy the same assumptions as $K(s, t)$ in* 1.1., *the constant A being independent of n and let us denote $\sigma_n(x, t) = (B) \int_a^b x(s)\, K_n(s, t)\, ds$. If there exists a set H dense in $B^M(\langle a, b\rangle; \mathcal{X})$ in which $\lim_{m, n \to \infty} \|\sigma_m(x, \cdot) - \sigma_n(x, \cdot)\|_M = 0$ or $\lim_{n \to \infty} \|\sigma_n(x, \cdot) - x(\cdot)\|_M = 0$, then the above relations hold everywhere in $B_M(\langle a, b\rangle; \mathcal{X})$.*

The proof by means of the triangle inequality is obvious.

1.3. Theorem. *If $M(u)$ fulfils the Δ_2-condition, $K_n(s, t)$ satisfy the same assumptions as in* **1.2.**, *and if for any $\langle \alpha, \beta\rangle \subset \langle a, b\rangle$, $\lim_{n\to\infty} \left\|\int_\alpha^\beta K_n(s, \cdot)\, ds - \chi_{\langle\alpha, \beta\rangle}(\cdot)\right\|_M = 0$, then*

$$\lim_{n \to \infty} \|\sigma_n(x, \cdot) - x(\cdot)\|_M = 0 \quad \textit{for each} \quad x(\cdot) \in B^M(\langle a, b\rangle; \mathcal{X}).$$

Proof. Let $a=\alpha_1<\beta_1=\alpha_2<\beta_2=\ldots<\beta_k=b$, and $x(s)=\sum_{j=1}^{k} x_j\,\chi_{\langle\alpha_j,\beta_j\rangle}(s)$, where $x_j\in\mathfrak{X}$. Then

$$\|\sigma_n(x,\cdot)-x(\cdot)\|_M\leqslant\sum_{j=1}^{k}\|x_j\|\left\|\int_{\alpha_j}^{\beta_j}K_n(s,\cdot)\,ds-\chi_{\langle\alpha_j,\beta_j\rangle}(\cdot)\right\|_M\xrightarrow[n\to\infty]{}0.$$

Since the set of all step-functions is dense in $B^M(\langle a,b\rangle;\mathfrak{X})$ ([2], p. 198), we have by virtue of **1.2**,

$$\lim_{n\to\infty}\|\sigma_n(x,\cdot)-x(\cdot)\|_M=0\quad\text{for}\quad x(\cdot)\in B^M(\langle a,b\rangle;X).$$

2.2. The classical theorem of M. Riesz on convergence in mean ([7], p. 266, Th. 6.4.) says that, if $f(\cdot)\in L^k\langle 0,2\pi\rangle$, $1<p<\infty$, then $\lim\limits_{n\to\infty}\|S_n(f,\cdot)-f(\cdot)\|_p=0$. Now we show that this theorem cannot be extended to the case of vector-valued functions, yet the analogy of classical Jackson and Quade theorems for vector-valued functions is true.

2.1. Counter-example. *There exists a continuous vector-valued function $x(s)$ from $\langle 0,2\pi\rangle$ to $L^1\langle 0,2\pi\rangle$ such that*

$$\lim_{n\to\infty}\|S_n(x,\cdot)-x(\cdot)\|_p=\infty \textit{ for any } p>0.$$

Proof. It is known that there exists a function $f(\cdot)\in L^1\langle 0,2\pi\rangle$ for which $\lim\limits_{n\to\infty}\|S_n(f,\cdot)\|_1=\infty$ ([7], p. 185). Evidently, the vector-valued function $x(s)=f(s+t)$ with values in $L^1\langle 0,2\pi\rangle$ is continuous ([3] p. 46, Th. 3.6.3.), consequently $x(\cdot)\in B^p_{2\pi}(L^1\langle 0,2\pi\rangle)$ for any $p>0$. Since

$$c_m(x)=\frac{1}{2\pi}(B)\int_0^{2\pi}x(s)\,e^{-ims}\,ds=\frac{1}{2\pi}\int_0^{2\pi}f(s+t)\,e^{-ims}\,ds=$$

$$=e^{imt}\,\frac{1}{2\pi}\int_0^{2\pi}f(s)\,e^{-ims}\,ds=e^{imt}\,c_m(f),$$

we have

$$S_n(x,s)=\sum_{m=-n}^{n}c_m(x)\,e^{ims}=\sum_{m=-n}^{n}c_m(f)\,e^{im(s+t)},$$

$$\|S_n(x,s)\|_1=\int_0^{2\pi}\left|\sum_{m=-n}^{n}c_m(f)\,e^{im(s+t)}\right|dt=\int_0^{2\pi}\left|\sum_{m=-n}^{n}c_m(f)\,e^{ims}\right|ds=\int_0^{2\pi}\left|S_n(f,s)\right|ds$$

and finally we have

$$\varliminf_{n\to\infty}\int_0^{2\pi}\|S_n(x,s)\|_1^p\,ds\geqslant\int_0^{2\pi}\varliminf_{n\to\infty}\|S_n(x,s)\|_1^p\,ds=\infty.$$

2.2. Lemma. *Let for any measurable function $f(s)$ defined in $\langle a,b\rangle$, $\|f(\cdot)\|_M=\sup\limits_{\rho(g,N)\leqslant 1}\int_a^b|f(s)\,g(s)|\,ds$ ($\|f(\cdot)\|_M$ may be infinity). If $K(s,t)$ is measurable in $\langle a,b\rangle\times\langle c,d\rangle$, then $\left\|\int_a^b|K(s,\cdot)|\,ds\right\|_M\leqslant\int_c^b\|K(s,\cdot)\|_M\,ds$.*

Proof.

$$\left\|\int_a^b |K(s,\cdot)|\,ds\right\|_M = \sup_{\rho(g,N)\leqslant 1}\int_c^d \left|\left[\int_a^b |K(s,t)|\,ds\right] g(t)\right| dt \leqslant$$

$$\leqslant \int_a^b \left[\sup_{\rho(g,N)\leqslant 1}\int_c^d |K(s,t)\,g(t)|\,dt\right] ds = \int_a^b \|K(s,\cdot)\|_M\,ds.$$

2.2.1. Corollary. *For* $M(u)=u^p$, $1<p<\infty$, *we obtain the integral Minkowski inequality*

$$\left\{\int_c^d \left[\int_a^b |K(s,t)|\,ds\right]^p dt\right\}^{1/p} \leqslant \int_a^b \left\{\int_c^d |K(s,t)|^p\,dt\right\}^{1/p} ds.$$

2.3 Theorem. *Let*

$$x(\cdot)\in B_{2\pi}^M(\mathcal{X}),\qquad w_n(s)=\frac{3}{2\pi n(2n^2+1)}(B)\int_0^{2\pi} x(t)\left[\frac{\sin n\frac{t-s}{2}}{\sin\frac{t-s}{2}}\right]^4 dt;$$

then

$$1)\ w_n(s)=A+\sum_{k=1}^{2n-3}(a_k\cos ks+b_k\sin ks),\ A,a_k,b_k\in\mathcal{X},$$

$$2)\ \|w_n(\cdot)-x(\cdot)\|_M\leqslant b\,\omega_M\left(\frac{1}{n}\right).$$

Proof. By means of the inequality $\omega_M(\lambda\delta)\leqslant(\lambda+1)\,\omega_M(\delta)\ (\lambda>0)$ and by virtue of **2.2.** one may proceed similarly as in the classical case ([5], p. 106—122).

2.4. Lemma. *Let* $w_n(s)$ *be any trigonometric polynomial with degree not higher than* n *and coefficients belonging to* X. *Then*

$$\|S_n(x,\cdot)-x(\cdot)\|_M = O(\|w_n(\cdot)-x(\cdot)\|_M\log n).$$

Proof. Since $w_n(s)-S_n(x,s)=\frac{1}{\pi}(B)\int_0^{2\pi}[w_n(s+t)-x(s+t)]\,D_n(t)\,dt$, where $D_n(t)$ is the Dirichlet kernel, we have from **2.2.**

$$\|w_n(\cdot)-S_n(x,\cdot)\|_M\leqslant\|w_n(\cdot)-x(\cdot)\|_M\cdot\frac{1}{\pi}\int_0^{2\pi} D_n(t)\,dt =$$

$$= O(\|w_n(\cdot)-x(\cdot)\|_M\log n),$$

or

$$\|S_n(x,\cdot)-x(\cdot)\|_M = O(\|w_n(\cdot)-x(\cdot)\|_M\log n)$$

2.5. Theorem. *If* $x(\cdot)\in Lip(\alpha,M)$ (*viz.* $\|x(\cdot+h)-x(\cdot)\|_M\leqslant L|h|^\alpha$, *where* L *is a constant*), $0<\alpha\leqslant 1$, *then* $\|S_n(x,\cdot)-x(\cdot)\|_M = O(n^{-\alpha}\log n)$.

Proof is obtained by virtue of **2.3.** and **2.4.**

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MATHEMATICS

On the Value of a Distribution at a Point

by

J. MIKUSIŃSKI

Presented by A. MOSTOWSKI on July 18, 1960

1. After S. Łojasiewicz [1], [2], the value of a distribution at a point x_0 is defined as the distributional limit

$$\lim_{h\to 0} f(x_0+hx) \tag{1}$$

provided that such limit exists. If it exists it is always a constant function [4]. The value $f(x_0)$ of $f(x)$ at x_0 is defined as the value of this constant function.

The distribution $f(x)$ may be one dimensional or q-dimensional with $q>1$. In the case when $q>1$, x_0, x and h considered above are vectors in the q-dimensional space

$$x_0=(\xi_{01},\ldots,\xi_{0q}),\qquad x=(\xi_1,\ldots,\xi_q),\qquad h=(\chi_1,\ldots,\chi_q);$$

by x^k, where $k=(\varkappa_1,\ldots,\varkappa_q)$, we then understand the $(\varkappa_1+\ldots+\varkappa_q)$-th power of the length $|x|$ of x, the symbol $h\to 0$ means that the length of h tends to 0.

Łojasiewicz has also considered, in the case $q>1$, other non-equivalent definitions of the value, which will not be considered here.

The value $f(x_0)$ of a function is an irregular operation [3] performed on functions. This operation can be extended to an operation on distributions $f(x)$, and this gives an alternative definition of the value of a distribution. We say that a distribution $f(x)$ has a value at x_0, if the ordinary limit

$$\lim_{h\to\infty}\int_{-\infty}^{\infty} f(x_0-t)\,\delta_n(t)\,dt \tag{2}$$

exists for every sequence of smooth (indefinitely derivable) functions $\delta_n(x)$ such that

1° $\delta_n(x)=0$ for $|x|>\varepsilon_n$, where $\varepsilon_n>0$, $\varepsilon_n\to 0$;

2° $\displaystyle\int_{-\infty}^{\infty}\delta_n(x)\,dx=1$;

3° $|x^k\,\delta^{(k)}(x)|<M_k\,\varepsilon_n^{-q}$ (M_k independent of n).

The value of the limit (2) is meant as the value $f(x_0)$ of the distribution $f(x)$ at x_0.

The purpose of this paper is to prove that both definitions of the value mentioned above are equivalent.

2. Suppose first that the limit (1) exists. We shall prove then that limit (2) also exists and that both are equal. Without loss of generality we may admit $x_0 = 0$ and $f(x_0) = 0$. By a necessary and sufficient condition of Łojasiewicz ([1], p. 7), there exists a continuous function $F(x)$ and an order k such that $F^k(x) = f(x)$ and

$$\lim_{h\to 0} \frac{F^{(k)}(h)}{h^k} = 0,$$

if $q > 1$, the symbol $F^{(k)}(x)$ denotes the distributional derivative

$$\frac{\partial^{\varkappa_1+\ldots+\varkappa_q}}{\partial\xi_1^{\varkappa_1}\ldots\partial\xi_q^{\varkappa_q}} F(x),$$

where $k = (\varkappa_1, \ldots, \varkappa_q)$. Thus, for any number $\varepsilon > 0$, there is a number $\eta > 0$ such that

$$\left|\frac{F^{(k)}(h)}{h^k}\right| < \varepsilon \quad \text{for} \quad |h| < \eta,$$

where $|h|$ denotes the length of h.

For sufficiently great values of n we have $\varepsilon_n < \eta$ and, consequently,

$$\left|\int_{-\infty}^{\infty} F(-t)\,\delta_n(t)\,dt\right| = \left|\int_{|x|<\varepsilon_n} \frac{F(-t)}{t^k}\, t^k(t)\,dt\right| \leq M_k\,\varepsilon_n^{-q} \int_{|x|<\varepsilon_n} |t^{-k}F(-t)|\,dt < \varepsilon M_k,$$

which proves that the limit (2) (with $x_0 = 0$) exists and is 0.

3. Suppose, now, that the limit (2) exists. We shall admit again that $x_0 = 0$ and $f(x_0) = 0$. Let us consider first the one-dimensional case $q = 1$ for the sake of simplicity; then x and h are real variables. We put

$$\delta_n(x) = \frac{1}{h_n}\,\Delta\!\left(\frac{x}{h_n}\right), \tag{3}$$

where $\Delta(x)$ is any smooth function such that $\Delta(x) = 0$ outside a bounded interval and $\int_{-\infty}^{\infty} \Delta(x)\,dx = 1$, $h_n > 0$, $h_n \to 0$. The sequence $\delta_n(x)$ satisfies conditions 1°—3°, which is easily verified. Since the limit is 0 for any sequence ε_n, such that $\varepsilon_n > 0$, $\varepsilon_n \to 0$ we also have

$$\lim_{h\to 0} \int_{-\infty}^{\infty} f(-t)\,\frac{1}{h}\,\Delta\!\left(\frac{t}{h}\right) dt = 0,$$

or, after substitution $t = hx$,

$$\lim_{h\to 0} \int_{-\infty}^{\infty} f(-hx)\,\Delta(x)\,dx = 0.$$

Since this holds for every smooth function $\Delta(x)$ which vanishes outside a bounded interval, this means that the limit (1) (with $x_0 = 0$) exists and is 0.

In order to extend the proof given above to an arbitrary q, it suffices to assign to the symbols a more general interpretation.

4. A remark should be added here. In (2) it suffices to suppose that the limit exists for every sequence $\delta_n(x)$ satisfying conditions 1°—3°. This implies that the limit is unique (i.e. it does not depend on the choice of $\delta_n(x)$). If, instead of 1°—3°, we only suppose that (2) holds for sequences $\delta_n(x)$ of the form (3), the uniqueness of the limit is not ensured. However, if the uniqueness of (3) is added as a supplementary supposition, conditions 1°—3° can be equivalently replaced by (3). By this slight modification we obtain a third equivalent definition of the value of a distribution.

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MATHEMATICS

The Determinant Theory of the Carleman Type

by

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Let X be a Hilbert space, Ξ — the space of all linear bounded functionals on X. The letters x, y, z (with indices) denote elements of X, and ξ, η, ξ — elements of Ξ. ξx is the value of ξ at x. $\mathfrak{E}_0$ is the Banach space of scalars. For $n>0$, $\mathfrak{E}_n$ is the Banach space of all bounded $2n$-linear functionals B on $\Xi^n \times X^n$ with the ordinary norm. $B\begin{pmatrix}\xi_1, \ldots, \xi_n\\ x_1, \ldots, x_n\end{pmatrix}$ denotes the value of $B \in \mathfrak{E}_n$ at $(\xi_1, \ldots, \xi_n, x_1, \ldots, x_n) \in \Xi^n \times X^n$. In particular, $\mathfrak{E}_1$ is the Banach space of bounded bilinear functionals on $\Xi \times X$, called operators. If $A \in \mathfrak{E}_1$, we write also $\xi A x$ instead of $A\begin{pmatrix}x\\ \xi\end{pmatrix}$. Every $A \in \mathfrak{E}_1$ can be interpreted as an endomorphism in X (in a natural way); Ax denotes the value of this endomorphism at x. Every $A \in \mathfrak{E}_1$ can also be interpreted as an endomorphism in Ξ; ξA denotes the value of this endomorphism at ξ. Obviously, $y = Ax$ and $\eta = \xi A$ are adjoint endomorphisms. $\mathfrak{E}_1$ is a Banach algebra (interpret $A \in \mathfrak{E}_1$ as endomorphisms!) with the unit I: $\xi I x = \xi x$, $Ix = x$, $\xi I = \xi$. For fixed ξ_0, x_0, $x_0 \cdot \xi_0$ denotes the one-dimensional operator K: $\xi K x = (\xi x_0) \cdot (\xi_0 x)$. $\mathfrak{Z}$ denotes the set of all operators which, if X is interpreted as l^2, are determined by an infinite square matrix $(\tau_{i,j})$ such that

$$\|T\| = \sqrt{\sum_{i,j} |\tau_{i,j}|^2} < \infty.$$

$\mathfrak{Z}$ is a Banach algebra with the norm $\|\;\|$, and an ideal in $\mathfrak{E}_1$. If $S = T_1 T_2$, where $T_1, T_2 \in \mathfrak{Z}$, then the trace $\operatorname{tr} S$ of S is well defined (viz. $\operatorname{tr} S = \sum_{i,j} \tau_{i,j}\sigma_{j,i}$, where $(\tau_{i,j})$ and $(\sigma_{i,j})$ are infinite square matrices representing T_1 and T_2, respectively, X being interpreted as l^2). If $T \in \mathfrak{Z}$ and $B \in \mathfrak{E}_n$ is such that, for fixed $\xi_1, \ldots, \xi_{n-1}$, $x_1, \ldots, x_{n-1}$, there exists an $S \in \mathfrak{Z}$ such that $\xi_n S x_n = B\begin{pmatrix}\xi_1, \ldots, \xi_n\\ x_1, \ldots, x_n\end{pmatrix}$, we write

$$T_{\xi_n x_n} B\begin{pmatrix}\xi_1, \ldots, \xi_n\\ x_1, \ldots, x_n\end{pmatrix}$$

instead of tr TS. Clearly, this expression is a linear function of each of the variables $\xi_1, \ldots, \xi_{n-1}, x_1, \ldots, x_{n-1}$, but does not depend on the bound variables ξ_n, x_n. If $D(T)$ is an analytic mapping from $\mathfrak{S}$ into $\mathfrak{D}_m$ $(m = 0, 1, 2, \ldots)$, then

$$D^{(1)}(T; T_1)\, D'(T; T_1) = \lim_{\varepsilon \to 0} \varepsilon^{-1} (D(T+\varepsilon T_1) - D(T)),$$

and by induction

$$D^{(n)}(T; T_1, \ldots, T_n) = \lim_{\varepsilon \to 0} \varepsilon^{-1} (D(T+\varepsilon T_n; T_1, \ldots, T_{n-1}) - D(T; T_1, \ldots, T_{n-1}))$$

for $T, T_1, \ldots, T_n \in \mathfrak{S}$.

For every $T \in \mathfrak{S}$, let T_n^m be the $2n$-linear functional

$$T_n^m \begin{pmatrix} \xi_1, \ldots, \xi_n \\ x_1, \ldots, x_n \end{pmatrix} = \sum_{\substack{i_1 + \ldots + i_n = m \\ 0 \leqslant i_1, \ldots, i_n}} \det(\xi_j T^{i_j} x_k) - \sum_{\substack{i_1 + \ldots + i_n = m \\ 0 < i_1, \ldots, i_n}} \det(\xi_j T^{i_k} x_k).$$

Obviously, $T_n^m \in \mathfrak{D}_n$. Let $D_{0,0}(T) = 1$ and, for $m = 1, 2, \ldots$,

$$D_{0,m}(T) = \begin{vmatrix} 0 & m-1 & 0 & 0 & . & 0 & . & . & . & . & 0 \\ \operatorname{tr} T^2 & 0 & m-2 & 0 & . & 0 & . & . & . & . & 0 \\ \operatorname{tr} T^3 & \operatorname{tr} T^2 & 0 & m-3 & & 0 & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ \operatorname{tr} T^{m-1} & \operatorname{tr} T^{m-2} & . & . & . & . & & \operatorname{tr} T^2 & 0 & & 1 \\ \operatorname{tr} T^m & \operatorname{tr} T^{m-1} & . & . & . & . & & \operatorname{tr} T^3 & \operatorname{tr} T^2 & & 0 \end{vmatrix}.$$

Let, for $n = 1, 2, \ldots$ and $m = 0, 1, 2, \ldots$,

$$D_{n,m}(T) = \left| \begin{array}{c|c} T_n^0 & n \quad 0 \quad \ldots \quad 0 \\ \hline T_n^1 & \\ T_n^2 & \\ \vdots & D_{0,m}(T) \\ T_n^m & \end{array} \right|,$$

Clearly, $D_{n,m}(T) \in \mathfrak{D}_n$ $(n, m = 0, 1, 2, \ldots)$, and

$$D_{n,m}(T) \begin{pmatrix} \xi_1, \ldots, \xi_n \\ x_1, \ldots, x_1 \end{pmatrix} = T_{\xi_{n+1} x_{n+1}} T_{\xi_{n+2} x_{n+2}} \cdots$$

$$T_{\xi_{n+m} x_{n+m}} \begin{vmatrix} \xi_1 x_1 & \ldots \xi_1 x_n & \xi_1 x_{n-1} & \xi_{1\, n+2} & \ldots \xi_1 x_{n+m} \\ . & . & . & . & . \\ \xi_n x_1 & \ldots \xi_n x_n & \xi_n x_{n+1} & \xi_n x_{n+2} & \ldots \xi_{n+2} x_{n+m} \\ \xi_{n+1} x_1 & \ldots \xi_{n+1} x_n & 0 & \xi_{n+1} \xi_{n+2} & \ldots \xi_{n+2} x_{n+m} \\ \xi_{n+2} x_1 & \ldots \xi_{n+2} x_n & \xi_{n+2} x_{n+1} & 0 & \ldots \xi_{n+2} x_{n+m} \\ . & . & . & . & . \\ \xi_{n+m} x_1 & \ldots \xi_{n+m} x_n & \xi_{n+m} x_{n+1} & \xi_{n+m} x_{n+2} \ldots & 0 \end{vmatrix}.$$

(i) *For every* $T \in \mathfrak{S}$, *the series*

$$D_0(T) = \sum_{m=0}^{\infty} \frac{1}{m!} D_{0,m}(T)$$

converges absolutely. For $n > 0$, *the series*

$$D_n(T) = \sum_{m=0}^{\infty} \frac{1}{m!} D_{n,m}(T)$$

of $2n$-*linear functionals converges in norm in the Banach space* $\mathfrak{D}_n$. *Moreover,*

$$|D_n(T)| \leqslant 2^n (n+m)^{\frac{n+m}{2}} (2\sqrt{e})^{n+m} \|T\|^m.$$

The first part of Theorem (i) is known: $D_0(T)$ is the Carleman [1] — Smithies [3] determinant of the operator $A = I+T$.

(ii) *For every* $T \in \mathfrak{S}$, *the sequence* $D_0(T), D_1(T), D_2(T), \ldots$ *is a determinant system for the operator* $A = I+T$ *in the sense defined by Sikorski* [2].

In particular, there exists an integer r such that $D_r(T) \neq 0 \in \mathfrak{D}_r$. Let $r(T)$ be the smallest integer r with this property.

(iii) *Let* $T \in \mathfrak{S}$, $r = r(T)$, *and let* $\eta_1, \ldots, \eta_r, y_1, \ldots, y_r$ *be fixed elements such that* $\delta = D_r(T)\begin{pmatrix}\eta_1, \ldots, \eta_r\\ y_1, \ldots, y_r\end{pmatrix} \neq 0$. *Then there exist elements* $\zeta_1, \ldots, \zeta_r, z_1, \ldots, z_r$ *and* $B \in \mathfrak{D}_1$ *such that, for all* ξ, x,

$$\zeta_i x = D_r(T)\begin{pmatrix}\eta_1, & \ldots\ldots\ldots & , \eta_r\\ y_1, \ldots, x_{i-1}, x, y_{i-1}, \ldots, y_r\end{pmatrix},$$

$$\xi z_i = D_r(T)\begin{pmatrix}\eta_1, \ldots, \eta_{i-1}, \xi, \eta_{i+1}, \ldots, \eta_r\\ y_1, & \ldots\ldots\ldots & , y_r\end{pmatrix},$$

$$\xi B x = \delta^{-1} D_{r+1}(T)\begin{pmatrix}\xi, \eta_1, \ldots, \eta_r\\ x, y_1, \ldots, y_r\end{pmatrix}.$$

The elements $\zeta_1, \ldots, \zeta_r$ *are linearly independent in* Ξ, *and so are* $z_1, \ldots, z_r$ *in* X.

The equation

$$x + Tx = x_0$$

has a solution x *if and only if* $\zeta_i x_0 = 0$ *for* $i = 1, \ldots, r$. *Then the general form of the solution* x *is* $x = Bx_0 + c_1 z_1 + \ldots + c_r z_r$.

The adjoint equation

$$\xi + \xi T = \xi_0$$

has a solution ξ *if and only if* $\xi_0 z_i = 0$ *for* $i = 1, \ldots, r$. *Then the general form of the solution* ξ *is* $\xi = \xi_0 B + c_1 \zeta_1 + \ldots + c_r \zeta_r$.

In the case of $r = 0$, Theorem (iii) asserts that $(I+T)^{-1} = D_1(T)/D_0(T)$.

For every $T \in \mathfrak{S}$, let $D_0^*(T) = D_0(T)$, and for $n > 0$ let $D_n^*(T)$ be the $2n$-linear functional

$$D_n^*(T)\begin{pmatrix}\xi_1, \ldots, \xi_n\\ x_1, \ldots, x_n\end{pmatrix} = D_n(T)\begin{pmatrix}\xi_1 T, \ldots, \xi_n T\\ x_1, \ldots, x_n\end{pmatrix} = D_n(T)\begin{pmatrix}\xi_1, \ldots, \xi_n\\ Tx_1, \ldots, Tx_n\end{pmatrix}.$$

The expression $D_1^*(T) = TD_1(T) = D_1(T)T$ is the Carleman [1] — Smithies [3] subdeterminant of $A = I+T$. More generally, $D_n^*(T)$ $(n > 0)$ are the abstract analogues of the Carleman [1] subdeterminants of A.

(iv) *Let $T \in \mathfrak{S}$. The smallest integer r such that $D_r^*(T) \neq 0 \in \mathfrak{D}_r$ is equal to $r(T)$, and $D_r^*(T) = (-1)^r D_r$. Let $\eta_1, \ldots, \eta_r, y_1, \ldots, y_r$ be fixed elements such that $\delta = D_r^* \begin{pmatrix} \eta_1, \ldots, \eta_r \\ y_1, \ldots, y_r \end{pmatrix} \neq 0$. Then there exist elements $\zeta_1, \ldots, \zeta_r$, $z_1, \ldots, z_r$ and $B \in \mathfrak{S}$ such that, for all ξ, x,*

$$\zeta_i x = D_r^*(T) \begin{pmatrix} \eta_1, \ldots\ldots\ldots, \eta_r \\ y_1, \ldots, y_{i-1}, x, y_{i+1}, \ldots y_r \end{pmatrix},$$

$$\xi z_i = D_r^*(T) \begin{pmatrix} \eta_1, \ldots, \eta_{i-1}, \xi, \eta_{r+1} \ldots, \eta_r \\ y_1, \ldots\ldots\ldots, y_r \end{pmatrix},$$

$$\xi B x = \delta^{-1} D_{r+1}^*(T) \begin{pmatrix} \xi, \eta_1, \ldots, \eta_r \\ x, y_1, \ldots, y_r \end{pmatrix}.$$

The elements $\zeta_1, \ldots, \zeta_r$ are linearly independent in Ξ, and so are $z_1, \ldots, z_r$ in X.

The equation

$$x + Tx = x_0$$

has a solution x if and only if $\zeta_i x_0 = 0$ for $i = 1, \ldots, r$. Then the general form of the solution x is $x = x_0 - Bx_0 + c_1 z_1 + \ldots + c_r z_r$.

The adjoint equation

$$\xi + \xi T = \xi_0$$

has a solution ξ if and only if $\xi_0 z_i = 0$ for $i = 1, \ldots, r$. Then the general form of the solution ξ is $\xi = \xi_0 - \xi_0 B + c_1 \zeta_1 + \ldots + c_r \zeta_r$.

In the case of $r = 0$, Theorem (iv) asserts that $(I+T)^{-1} = I - D_1^*(T)\, D_0^*(T)$.

(v) *The following identities hold*:

$$D_n(T) \begin{pmatrix} \xi_1, \ldots, \xi_n \\ x_1, \ldots, x_n \end{pmatrix} = D'_{n-1}(T; x_n \cdot \xi_n) \begin{pmatrix} \xi_1, \ldots, \xi_{n-1} \\ x_1, \ldots, x_{n-1} \end{pmatrix} + D_{n-1}(T) \cdot \begin{pmatrix} \xi_1, \ldots, \xi_{n-1} \\ x_1, \ldots, x_{n-1} \end{pmatrix} \cdot \xi_n x_n;$$

$$\begin{aligned} D_n(T) \begin{pmatrix} \xi_1, \ldots, \xi_n \\ x_1, \ldots, x_n \end{pmatrix} &= D_0^{(n)}(T; x_1 \cdot \xi_1, \ldots, x_n \cdot \xi_n) + \\ &+ \sum_{i=1}^{n} D_0^{(n-1)}(T; x_1 \cdot \xi_1, \ldots, x_{i-1} \cdot \xi_{i-1}, x_{i-1} \cdot \xi_{i+1}, \ldots, x_n \cdot \xi_n) \cdot \xi_i x_i + \\ &+ \sum_{\substack{i,j=1 \\ i<j}}^{n} D_0^{(n-2)}(T; x_1 \cdot \xi_1, \ldots, x_{i-1} \cdot \xi_{i-1}, x_{i+1} \cdot \xi_{i+1}, \ldots, x_{j-1} \cdot \\ &\qquad \cdot \xi_{j-1}, x_{j-1} \cdot \xi_{j-1}, \ldots, x_n \cdot \xi_n) \cdot \xi_i x_i \cdot \xi_j x_j + \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &+ \sum_{i=1}^{n} D'_0(T; x_i \cdot \xi_i) \cdot \xi_1 x_1 \cdot \ldots \cdot \xi_{i-1} x_{i-1} \cdot \xi_{i+1} x_{i+1} \cdot \ldots \cdot \xi_n x_n + \\ &+ D_0(T) \cdot \xi_1 x_1 \cdot \ldots \cdot \xi_n x_n; \end{aligned}$$

$$D_0^{(n)}(T; x_1 \cdot \xi_1, \ldots, x_n \cdot \xi_n) = D_n(T)\begin{pmatrix}\xi_1, \ldots, \xi_n \\ x_1, \ldots, x_n\end{pmatrix} -$$

$$- \sum_{i=1}^{n} D_{n-1}(T)\begin{pmatrix}\xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_n \\ x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\end{pmatrix} \cdot \xi_i x_i +$$

$$+ \sum_{\substack{i,j=1 \\ i<j}}^{n} D_{n-2}(T)\begin{pmatrix}\xi_1, \ldots, \xi_{i-1}, \xi_{i-1}, \ldots, \xi_{j-1}, \xi_{j-1}, \ldots, \xi_n \\ x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n\end{pmatrix} \cdot \xi_i x_i \cdot \xi_j x_j -$$

$$- \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$+ (-1)^{n-1} \sum_{i=1}^{n} D_1(T)\begin{pmatrix}\xi_i \\ x_i\end{pmatrix} \cdot \xi_1 x_1 \cdot \ldots \cdot \xi_{i-1} x_{i-1} \cdot \xi_{i+1} x_{i+1} \cdot \ldots \cdot \xi_n x_n +$$
$$+ (-1)^n D_0(T) \cdot \xi_1 x_1 \cdot \ldots \cdot \xi_n x_n.$$

(vi) *$D_0(T)$ is the only analytic function on the Banach space $\mathfrak{S}$ such that*

$$D_0'(T; (I+T)T_1) = -D_0(T) \cdot \operatorname{tr} TT_1 \qquad (T, T_1 \in \mathfrak{S}),$$

and $D_0(0) = 1$.

Proofs of Theorems (i)—(vi) will be published in the paper *On the Carleman determinants*, to appear in Studia Mathematica.

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MATHEMATICS

On Substitutions in the Dirac Delta Distribution

by

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If $f(y)$ is a distribution in an open subset O' of the p-dimensional space, and

$$\sigma(x) = (\sigma_1(x), \ldots, \sigma_p(x)) \quad (x = (\xi_1, \ldots, \xi_q))$$

is a transformation of an open subset O of the q-dimensional space into O', such that

(a) $p \leqslant q$,

(b) all the functions $\sigma_1(x), \ldots, \sigma_p(x)$ are infinitely derivable,

(c) the function

$$J(x) = \sqrt{\sum_{j_1 < \ldots < j_p} \left(\frac{\partial(\sigma_1, \ldots, \sigma_p)}{\partial(\xi_{j_1}, \ldots, \xi_{j_p})} \right)^2}$$

does not vanish in O,

then the substitution $f(\sigma(x))$ is feasible and it is a distribution in O, [2].

The simplest non-trivial case is where $p = q = 1$ and $f(y)$ is the one-dimensional Dirac delta distribution $\delta_1(y)$. We then have the formula (see e.g. [1], § 12)

$$\delta_1(\sigma(x)) = \sum_n |\sigma'(x_n)|^{-1} \delta_1(x - x_n),$$

i.e.

$$\delta_1(\sigma(x)) = \sum_n J(x_n)^{-1} \delta_1(x - x_n), \tag{1}$$

where $x_1, x_2, \ldots$ are all the points x such that $\sigma(x) = 0$ (if such points do not exist, the right-hand side is equal to zero).

The purpose of this paper is to generalize formula (1) over the case of arbitrary p and q, $1 \leqslant p \leqslant q$ (see [2]). We recall that the p-dimensional Dirac delta distribution is the direct product

$$\delta_p(y) = \delta_1(\eta_1) \ldots \delta_p(\eta_p) \quad (y = (\eta_1, \ldots, \eta_p)).$$

If the set

$$S = \{x : \sigma(x) = 0\}$$

is empty, then of course $\delta_p(\sigma(x))$ is the zero distribution.

In the sequel we shall investigate only the non-trivial case, where S is not empty, i.e. S is a $(q-p)$-dimensional surface lying in O. The formula (1) suggests that $\delta_p(\sigma(x))$ is a distribution of a mass on the surface S with the density $J(x)^{-1}$, i.e. that

(2) $$\delta_p(\sigma(x)) = \int_S J(t)^{-1}\, \delta_q(x-t)\, dt,$$

the surface integral being taken with respect to the $(q-p)$-dimensional area on S. We shall prove this formula.

Since $J(t)^{-1}\delta_q(x-t)$ is a $2q$-dimensional distribution (defined in the Cartesian product of O and the q-dimensional space) continuous in the variable t, the integral

$$\int_S J(t)^{-1}\, \delta_q(x-t)\, dt$$

exists for every $S_0 \subset S$ with a finite area [3] and is a distribution vanishing outside S_0. Decomposing S into a sequence of disjoint sets S_0 of finite areas, we infer that the integral

$$\int_S J(t)^{-1}\, \delta_q(x-t)\, dt$$

exists and is a distribution vanishing outside S. The distribution $\delta_p(\sigma(x))$ also vanishes outside S. Thus, the equality (2) holds in the open set $O \setminus S$. It remains to prove that (2) holds in a neighbourhood I of every point $x_0 \in S$.

Since $J(x_0) \neq 0$, one of the jacobians defining $J(x)$ does not vanish, say

$$\left(\frac{\partial(\sigma_1, \ldots, \sigma_p)}{\partial(\xi_1, \ldots, \xi_p)}\right)_{x=x_0} \neq 0.$$

Thus, the equations

$$\sigma_1(x) = 0, \ldots, \sigma_p(x) = 0$$

can be solved:

$$\xi_1 = \varphi_1(\xi_{p+1}, \ldots, \xi_q), \ldots, \xi_p = \varphi_p(\xi_{p+1}, \ldots, \xi_q)$$

in a neighbourhood of x_0, the functions $\varphi_1, \ldots, \varphi_p$ being derivable.

Let I be an open interval such that $x_0 \in I$, $\bar{I} \subset O$, and

$$S \cap I = \{x\colon \xi_1 = \varphi_1(\xi_{p+1}, \ldots, \xi_q), \ldots, \xi_p = \varphi_p(\xi_{p+1}, \ldots, \xi_q)\}$$

where $(\xi_1, \ldots, \xi_p)$ runs over the whole p-dimensional basis of I. Let

$$I_\varepsilon = \{x \in I\colon \ |\sigma_j(x)| \leqslant \varepsilon \quad \text{for} \quad j = 1, \ldots, p\}.$$

For any function $G(x,t)$ continuous in the neighbourhood of $\bar{I} \times \bar{I}$ we have

$$\lim_{\varepsilon\to 0} (2\varepsilon)^{-p} \int_{I_\varepsilon} G(x,t)\, dt = \int_{S\cap I} J(t)^{-1} G(x,t)\, dt,$$

where the first integral is taken with respect to the q-dimensional volume, and the second — with respect to the $(q-p)$-dimensional area on S. The convergence is uniform in $x \in I$.

By differentation with respect to x, we obtain

$$\lim_{\varepsilon \to 0} (2\varepsilon)^{-p} \int_{I_\varepsilon} g(x, t)\, dt = \int_{S \cap I} J(t)^{-1} g(x, t)\, dt \quad \text{in } I$$

for every distribution $g(x, t)$ defined on a neighbourhood of $I \times I$ and continuous in the variable t. The convergence is here understood in the distributional sense Since the distribution $g(x, t) = \delta_q(x - t)$ is continuous in t, we have

$$\lim_{\varepsilon \to 0} (2\,\varepsilon)^{-1} \int_{I_\varepsilon} \delta_q(x - t)\, dt = \int_{S \cap I} J(t)^{-1} \delta_q(x - t)\, dt \quad \text{in } I.$$

Now let

$$\delta_{p,\varepsilon}(y) = \begin{cases} (2\varepsilon)^{-1} & \text{if} \quad |\eta_j| \leqslant \varepsilon \quad \text{for} \quad j = 1, \ldots, p \\ 0 \text{ elsewhere.} \end{cases}$$

Then,

$$(2\varepsilon)^{-p} \int_{I_\varepsilon} \delta_q(x - t)\, dt = \delta_{p,\varepsilon}(\sigma(x)) \quad \text{in } I\,.$$

Since $\lim_{\varepsilon \to 0} \delta_{p,\varepsilon}(y) = \delta_p(y)$ distributionally, we have also, [2],

$$\lim_{\varepsilon \to 0} \delta_{p,\varepsilon}(\sigma(x)) = \delta_p(\sigma(x))$$

Consequently,

$$\delta_p(\sigma(x)) = \int_{I \cap S} J(t)^{-1} \delta_q(x - t)\, dt \quad \text{in } I\,.$$

Since

$$\int_{S \setminus I} J(t)^{-1} \delta_q(x - t)\, dt = 0 \quad \text{in I},$$

we get the formula (2) in I.

In the case $p = 1$, formula (2) can be written as follows:

(3) $$\delta_1(\sigma(x)) = \int_S |\text{grad}\, \delta(x)|^{-1} \delta_q(x - t)\, dt\,.$$

In the case $p = q$, it can be written:

(4) $$\delta_q(\sigma(x)) = \sum_n \left| \left(\frac{\partial(\sigma_1, \ldots, \sigma_q)}{\partial(\xi_1, \ldots, \xi_q)} \right)_{x = x_n} \right|^{-1} \delta(x - x_n)\,,$$

where $x_1, x_2, \ldots$ are all the points x such that $\sigma(x) = 0$.

Observe that the above proof of (2) is valid under the following conditions (b′), (c′) which are weaker than (b) and (c) respectively:

(b′) all the functions $\sigma_1(x), \ldots, \sigma_p(x)$ are continuous and, in a neighbourhood of $S = \{x: \sigma(x) = 0\}$, they have continuous derivatives of the first order;

(c′) $J(x)$ does not vanish on S.

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MATHÉMATIQUE

Sur un problème de E. Marczewski concernant les fonctions périodiques

par

J. S. LIPIŃSKI

Présenté par E. MARCZEWSKI le 24 août 1960

C. Ryll-Nardzewski a posé la question s'il existe une suite de nombres réels $\{a_n\}$ ayant la propriété suivante: pour toute suite bornée $\{b_n\}$ il existe une fonction continue et presque-périodique $f(x)$ telle que $f(a_n) = b_n$. E. Marczewski a modifié ce problème en demandant que la fonction $f(x)$ soit non seulement presque-périodique, mais périodique. Le problème de C. Ryll-Nardzewski a été résolu positivement par J. Mycielski; (pour la démonstration voir [1]). Je vais montrer dans la présente communication que le problème de E. Marczewski admet également une solution positive. En effet, on a le théorème suivant:

THÉORÈME. *Soit* $\delta_n > 0$ *et* $\sum_{n=1}^{\infty} \delta_n = c < +\infty$ *Supposons, en outre, que les termes de la suite* $\{a_n\}$ *satisfont à l'inégalité*

$$\frac{a_{n+1}}{a_{n+1}} \geqslant \frac{c+\delta_{n+2}}{\delta_{n+1}}. \tag{1}$$

Alors il existe, pour toute suite bornée $\{b_n\}$, *une fonction* $f(x)$ *continue, périodique et monotone par intervalles, telle que* $f(a_n) = b_n$.

Démonstration. Soit B l'ensemble des termes de la suite $\{b_n\}$. Posons $Y = \langle \inf b_n, \sup b_n \rangle$. L'ensemble $Y \setminus B$ est ouvert. S'il n'est pas vide, nous désignons par Ω_n ses composantes et faisons correspondre à chaque composante $\Omega_n = (\omega_n, \omega_n + |\Omega_n|)$ un nombre $\gamma_n > 0$ tel que $\Sigma \gamma_n = \frac{1}{2}\delta_1$. A chaque nombre b_n faisons correspondre l'intervalle fermé $\langle d_n, d_n + \delta_{n+1}\rangle$, où $d_n = \sum_{b_i < b_n} \delta_{i+1} + \sum_{\omega_i < b_n} \gamma_n$.

Si l'ensemble $Y \setminus B$ est vide, dans la définition de d_n n'interviendra que la somme des nombres δ_{i+1}. A chaque intervalle Ω_n nous faisons correspondre l'intervalle fermé $\langle e_n, e_n + \gamma_n \rangle$, où $e_n = \sum_{b_i < \omega_n} \delta_{i+1} + \sum_{\omega_i < \omega_n} \gamma_i$.

Je vais maintenant définir dans l'intervalle $\langle 0, c\rangle$ une fonction continue $\varphi(x)$ (Fig. 1).

Pour $x \in \langle d_n, d_n + \delta_{n+1} \rangle$ posons $\varphi(x) = b_n$. Soit ensuite $\varphi(e_n) = \omega_n$, $\varphi(e_n + \gamma_n) = \omega_n + |\Omega_n|$. La fonction $\varphi(x)$ se trouve ainsi définie aussi aux extrémités

des intervalles $\langle e_n, e_n+\gamma_n\rangle$. Aux points intérieurs de ces intervalles nous la définissons de telle sorte qu'elle soit linéaire dans chacun d'eux. Pour $t=\sum_{n=1}^{s}\gamma_n+\sum_{n=2}^{\infty}\delta_n$, où s désigne le nombre des composantes de l'ensemble $Y\setminus\overline{B}$, nous posons $\varphi(t)=\sup b_n$. Enfin $\varphi(0)=\inf b_n$. Il peut alors arriver que la fonction $\varphi(x)$ ne soit pas définie

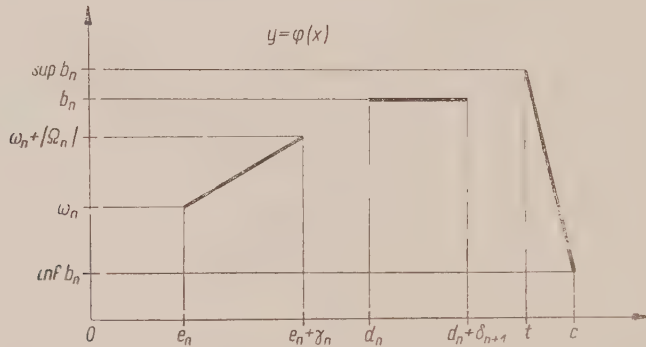

Fig. 1

en tout point de l'intervalle $\langle 0, t\rangle$, mais la somme des longueurs des intervalles $\langle d_n, d_n+\delta_{n+1}\rangle$ et $\langle e_n, e_n+\gamma_n\rangle$, dans lesquels la fonction $\varphi(x)$ est déjà définie, est égale à t. De plus, ces intervalles sont disjoints. La fonction $\varphi(x)$ est ainsi définie dans un sous-ensemble dense de l'intervalle $\langle 0, t\rangle$. On voit aisément qu'elle est non décroissante dans cet intervalle. En tout point $x\in(0, t)$ la fonction $\varphi(x)$ admet donc une limite à gauche et à droite. On doit avoir $\lim_{\mu\to x-0}\varphi(u)=\lim_{\mu\to x+0}\varphi(u)$, puisque l'ensemble des valeurs de la fonction déjà définies, c'est-à-dire l'ensemble $BU(U\,\overline{\Omega}_n)$, est dense dans Y. Donc, si la fonction $\varphi(x)$ n'a pas encore été définie en un tel point $x\in(0, t)$, nous admettrons en ce point $\varphi(x)=\lim_{\mu\to x}\varphi(u)$. On a aussi $\varphi(0)=$ $=\lim_{\mu\to 0+}\varphi(u)$ et $\varphi(t)=\lim_{\mu\to t-0}\varphi(u)$. Il est évident que la fonction ainsi définie dans l'intervalle $\langle 0, t\rangle$ est continue. Posons $\varphi(c)=\inf b_n$. La fonction $\varphi(x)$ est déjà définie aux extrémités de l'intervalle $\langle t, c\rangle$. Aux points intérieurs de cet intervalle définissons-la de telle sorte qu'elle soit linéaire dans tout l'intervalle $\langle t, c\rangle$. Nous obtenons ainsi finalement une fonction $\varphi(x)$ définie dans tout l'intervalle $\langle 0, c\rangle$, non décroissante dans l'intervalle $\langle 0, t\rangle$ et décroissante dans $\langle t, c\rangle$.

Posons $\Theta_n(r)=a_n/r$. Soit J_1 un intervalle fermé tel que $\Theta_1(J_1)=\langle d_1, d_1+\delta_2\rangle$. Posons encore $L_1=\Theta_2(J_1)$. Alors

$$J_1=\left\langle\frac{a_1}{d_1+\delta_2}, \frac{a_1}{d_1}\right\rangle \quad\text{et}\quad L_1=\left\langle\frac{a_2}{a_1}d_1, \frac{a_2}{a_1}(d_1+\delta_2)\right\rangle.$$

On constate aisément, en tenant compte de (1), que $|L_1|\geqslant c+\delta_3$. L'intervalle L_1 contient donc un intervalle de la forme $\langle k_2c+d_2, k_2c+d_2+\delta_3\rangle$, où k_2 est un nombre naturel. Désignons par J_2 un intervalle fermé tel que $\Theta_2(J_2)=\langle k_2c+$ $+d_2, k_2c+d_2+\delta_3\rangle\subset\Theta_2(J_1)$. Evidemment $J_1\supset J_2$. Supposons que nous avons

déjà défini, pour $i = 1, 2, \ldots, n-1$, les intervalles fermés J_i et les nombres naturels k_i tels que $J_i \supset J_{i+1}$ et $\Theta_i(J_i) = \langle k_i c+d_i,\ k_i c+d_i+\delta_{i+1}\rangle$. Soit $L_{n-1} = \Theta_n(J_{n-1})$. Posons $J_{n-1} = \langle r', r''\rangle$. Alors $r' = a_{n-1}(k_{n-1}c+d_{n-1}+\delta_n)^{-1}$, $r'' = a_{n-1}(k_{n-1}c+ +d_{n-1})^{-1}$ et on a

$$L_{n-1} = \left\langle \frac{a_n}{r''}, \frac{a_n}{r'} \right\rangle = \left\langle \frac{a_n}{a_{n-1}}(k_{n-1}c+d_{n-1}),\ \frac{a_n}{a_{n-1}}(k_{n-1}c+d_{n-1}+\delta_n) \right\rangle,$$

d'où

$$|L_{n-1}| = \frac{a_n}{a_{n-1}}\delta_n .$$

On trouve, d'après (1), que $|L_{n-1}| \geqslant c+\delta_{n+1}$. L'intervalle L_{n-1} contient donc un intervalle de la forme $\langle k_n c+d_n, k_n c+d_n+\delta_{n+1}\rangle$, où k_n est un nombre naturel. Désignons par J_n un intervalle fermé tel que $\Theta_n(J_n) = \langle k_n c+d_n, k_n c+d_n+\delta_{n+1}\rangle$. Evidemment $J_{n-1} \supset J_n$. Il existe un point r_0 tel que $r_0 \in \bigcap_{n=1}^{\infty} J_n$. Pour tout n on a

(2) $$\Theta_n(r_0) \in \langle k_n c+d_n,\ k_n c+d_n+\delta_{n+1}\rangle.$$

Définissons maintenant la fonction $f(x)$. La variable indépendante x peut être mise sous la forme $x = l_x cr_0+y_x$, où l_x est un nombre entier et y_x satisfait à l'inégalité $0 \leqslant y_x < cr_0$. Posons $f(x) = \varphi(y_x/r_0)$. La fonction $f(x)$ est continue et périodique, de période cr_0. Dans les intervalles $\langle \pm ncr_0,\ r_0 t \pm ncr_0\rangle$ elle est non décroissante et dans les intervalles $\langle r_0 t \pm ncr_0,\ (\pm n+1)cr_0\rangle$ elle est décroissante. Il reste à prouver que $f(a_n) = b_n$.

Soit

(3) $$a_n = l_{a_n} cr_0 + y_{a_n},$$

alors

(4) $$f(a_n) = \varphi\left(\frac{y_{a_n}}{r_0}\right).$$

En vertu de la définition de la fonction $\Theta_n(r)$ on a $\Theta_n(r_0) = r_0/a_n$. De la formule (3) on tire $\Theta_n(r_0) = l_{a_n}c + y_{a_n}/r_0$. On déduit de l'Eq. (2) que $k_n = l_{a_n}/r$ et $y_{a_n}/r \in \langle d_n, d_n+\delta_{n+1}\rangle$. D'après la définition de la fonction $\varphi(x)$ on a $\varphi(y_{a_n}/r_0) = b_n$. Cette égalité, rapprochée de (4), donne $f(a_n) = b_n$, ce qui achève la démonstration.

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THEORETICAL PHYSICS

Ferroelectric and Antiferroelectric Arrangements in Perovskite-Type Substances

by

A. JAŚKIEWICZ and H. KONWENT

Presented by W. RUBINOWICZ on August 6, 1960

In search for more advantageous solutions, various special dipole arrangements for a given crystal structure of ferroelectric substance have so far been studied (see, e.g. [1]). It was found that, taking only dipolė-dipole interactions into account, the antiferroelectric arrangement was preferable to the ferroelectric [2]. The present paper studies a dipole arrangement, which develops when a crystal transforms from a non-polar to a polar state. When transformation begins, one ion is displaced from its original position forming a dipole and producing an electric field in its surroundings. The effect of this field may be diverse depending on the arrangement of ions. Our starting point is not the dipole arrangement, but the crystal structure and its ionic polarizabilities. The local field method is not used, because in this method only the total polarization of a crystal is considered.

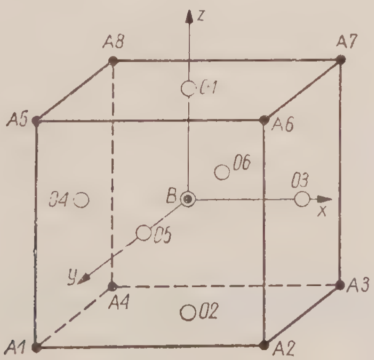

Fig. 1. Elementary cell of a perovskite-type cubic crystal

For the sake of simplicity, we shall discuss, as usually, the perovskite-type substances (ABO_3, Fig. 1).

Venevcev and Zhdanov [3] distinguish substances with various moving ions. In [3] it was assumed that there exist polar substances, where either the A or B ion is displaced.

The ABO_3 substance under consideration has a simple cubic lattice of A ions, face centered by O ions and body centered by B ions (Fig. 1). A thermal dipole is formed in an elementary cell, when a B ion is displaced from its original position. The thermal dipole induces elastic dipoles in the surrounding ions and acts on the nearest B ions tending to displace them. An electric field produced by the induced dipoles also acts at sites nearest to the B ions. We number the A and O ions belonging to the cell, where the primary displaced B ion is situated as in Fig. 1 and the B ions nearest to the primary cell as in Fig. 2. The electric field produced by the dipoles is calculated using the well known formula for the field of the point dipole

$$E = \frac{3r(rm) - r^2 m}{r^5}, \tag{1}$$

where r is the radius vector and m the dipole moment.

Let the primary dipole with an electric moment m be directed along the z-axis. Using Eq. (1) we calculate the field acting on A and O ions in the primary cell. Denoting polarizabilities of A and O ions by α_A and α_0 respectively, we obtain elastic dipoles induced in A and O ions by the primary thermal dipole.

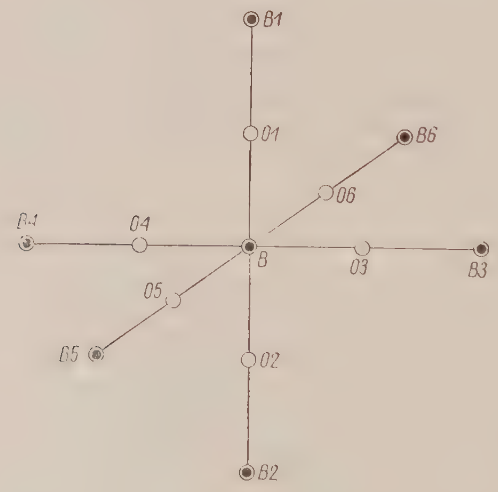

Fig. 2. Configuration of B ions nearest to the primary displaced B ion

On the site of a B ion in a neighbouring cell there acts an electric field produced by the primary thermal dipole and the elastic dipoles induced in the adjoining face (we use the approximation of the first neighbours). For example, on the site of the ion B1 (cf. Fig. 2), there acts the field produced by elastic dipoles induced in ions A5, A6, A7, A8 and 01, and by the thermal dipole.

Knowing the magnitude of the elastic dipoles we can write for the z-component of the electric field acting on ions B1 or B2

$$E_z^1 = \left(\frac{256}{a^3}\,\alpha_0 - \frac{512}{27a^3}\,\alpha_A + 2\right)\frac{m}{a^3}, \tag{2}$$

and for the z-component of the electric field acting on ions B3, B4, B5 or B6

$$E_z^3 = \left(\frac{256}{a^3}\,\alpha_0 - 1\right)\frac{m}{a^3}, \tag{3}$$

where a is the lattice constant. The x- and y-components of the electric field at the B-ion sites equal zero.

Since the displacement of B ions takes place under the action of an electric field, the sign of the field determines the direction of translation of B ions and consequently also the arrangement of thermal dipoles.

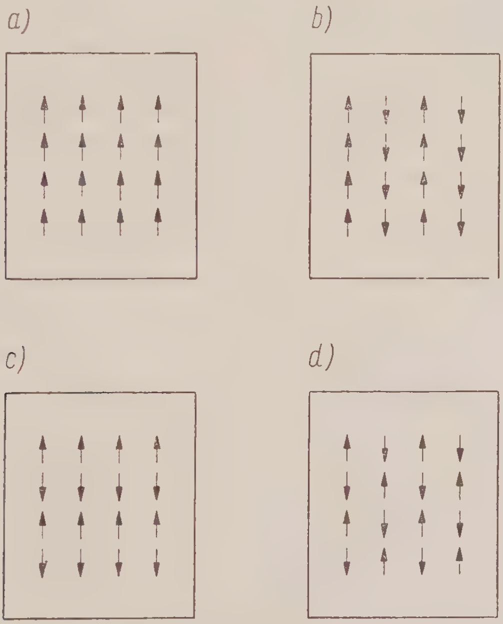

Fig. 3. Possible thermal dipole configurations for a perovskite crystal, when the B ion is displaced from its original position. *a)* $E_z^1 > 0$, $E_z^3 > 0$, *b)* $E_z^1 > 0$, $E_z^3 < 0$, *c)* $E_z^1 < 0$, $E_z^3 > 0$, *d)* $E_z^1 < 0$, $E_z^3 < 0$.

The signs of E_z^1 and E_z^3 determine the type of the dipole arrangement in a polar substance. When $E_z^1 > 0$, $E_z^3 > 0$ we have a ferroelectric arrangement. In all other cases we obtain three types of antiferroelectric arrangements (Fig. 3). Three configurations 3a, 3b, 3d were discussed by Kinase [4] using the local field method.

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THEORETICAL PHYSICS

Approximative Formulas for the Diffracted Electromagnetic Wave. I.

by

B. KARCZEWSKI

Presented by W. RUBINOWICZ on August 6, 1960

The correct formulation of Huygens' principle for electromagnetic waves is due to F. Kottler [4], whose theory constituted the natural termination of the road along which Larmor and Tedone had been working towards a solution of the problem. F. Kottler owed his success to the circumstance that he operated with electric and magnetic charges on the edge Γ of the diffracting aperture; this was a step forward with respect to the ideas of Larmor and Tedone, who considered the diffracted image to be produced by electric and magnetic charges and currents distributed in an appropriate manner over the surface S of the diffracting body. If S is a closed surface, then, clearly, Kottler's edge effect vanishes and we have the Larmor-Tedone formulas strictly. Now, what is the specific reason for the correctness of Kottler's formulas? Obviously, it is due to the fact that the solution proposed by Kottler for diffraction problems satisfies the full system of Maxwell's equations. Formulas derived within the framework of any diffraction theory should fulfil the Maxwell's system, this being a necessary condition for the correctness of the theory. Clearly, however, this is no sufficient condition. Such is precisely the case of Kottler's formulas. Kottler had solved the diffraction problem as a "saltus problem", whereas, if correctly formulated, it is a problem of boundary conditions. Kottler's formulas can be represented in the form of surface integrals taken over the surface of the diffracting aperture and curvilinear integrals along its edge Γ. The surface integrals imply a non-perturbed electric field, just as if the presence of the diaphragm did not affect the value of the field on the surface of the diffracting aperture, which, from the point of view of physics, is incorrect. Thus, the situation resembles that of Kirchhoff's scalar theory. Nevertheless it is a well-known fact that Kirchhoff's theory yields the correct distribution of the light intensity near the boundary of geometrical shadow. Hence, Kottler's theory should also be expected to yield results in agreement with experiment in the neighbourhood of geometrical shadow, satisfactorily approximating those of Kirchhoff's theory. Otherwise, Kottler's formulas would be incorrect, as they would fail to yield a quantitatively correct description of the phenomena of diffraction.

It is the aim of the present paper to provide an answer to the foregoing problem.

Kottler's theory

Assume an electric dipole to oscillate at point L, its moment being parallel to the z-axis. Let $\vec{E}$ and $\vec{H}$ denote the electric and magnetic field strength, respectively. Kottler's theory yields the following formulas for the values of the vectors $\vec{E}$ and $\vec{H}$ at P:

$$(1.1)\quad \begin{cases} \vec{E}(P) = \vec{E}^*(P) + \dfrac{1}{4\pi}\,\mathrm{grad}_L \dfrac{\partial}{\partial z_L} u^B(L,P) + \vec{\varkappa}\,\dfrac{k^2}{4\pi} u^B(L,P) + \\ \qquad + \dfrac{1}{4\pi}\displaystyle\int_\Gamma (\vec{ds}\times\vec{E}_0)\,w(Q,P) + \dfrac{1}{4\pi ik}\,\mathrm{grad}_P \int_\Gamma (\vec{ds}\,\vec{H}_0)\,w(Q,P) \\ \vec{H}(P) = \vec{H}^*(P) - \dfrac{ik}{4\pi}\,[\mathrm{grad}\, u_L^B(L,P)\times\vec{\varkappa}] + \\ \qquad + \dfrac{1}{4\pi}\displaystyle\int_\Gamma (\vec{ds}\times\vec{K}_0)\,w(Q,P) - \dfrac{1}{4\pi ik}\int_L (\vec{ds}\,\vec{E}_0)\,w(Q,L), \end{cases}$$

where

$$\vec{E}^*(L,P) = \vec{E}_0(L,P) = \frac{1}{4\pi}\,\mathrm{grad}_P \frac{\partial}{\partial z_P} u_0(L,P) + \vec{\varkappa}\,\frac{k^2}{4\pi} u_0(L,P)$$

$$\vec{H}^*(L,P) = \vec{H}_0(L.\,P) = \frac{ik}{4\pi}\,[\mathrm{grad}\, u_0(L,P)\times\vec{\varkappa}],$$

within the cone of light, and

$$\vec{E}^*(L,P) = \vec{H}^*(L,P) = 0$$

in the shadow;

$$(1.2)\qquad u^B((L,P) = \frac{1}{4\pi}\int_\Gamma \frac{e^{-ik(\rho+r)}}{r\varrho}\;\frac{(\vec{r}\times\vec{\varrho})\,\vec{t}}{\varrho r+\vec{\varrho}\,\vec{r}}\,ds$$

and

$$w(Q,P) = \frac{e^{\,ikr_{PQ}}}{r_{PQ}};$$

$\vec{\varkappa}$ denotes the unit vector in the direction of the z-axis, $\vec{t}$ — the one tangent to the curve Γ, r — the distance from the point of observation P to the current point Q on the edge Γ of the diffracting screen, ϱ — the distance from the source L to Q, and finally $u_0(L,P) = e^{-ikR}/R$, with R denoting the distance from L to P.

The terms in Eq. (1.1) containing the function $u^B(L,P)$ will be termed "Kirchhoffian" as this function given by Eq. (1.2) determines the diffracted wave of Kirchhoff's theory.

It will be remembered that Rubinowicz [2] was the first to transform Kirchhoff's surface integral into a curvilinear integral, thus materializing ideas due to Young, according to which the diffracted image results from interference of waves "reflected" from the edge of the screen and the incident wave. Kottler's formulas (1.1) can be considered to represent the formal expression of Young's ideas as adapted to electromagnetic waves.

Method of approximation

Clearly, all integrals in Eq. (1.1) can be reduced to one of the type

$$\int_{\Gamma} e^{-ik\zeta} g(s)\, ds, \tag{2.1}$$

where $\zeta = r+\varrho$, and $g(s)$ is a function of the parameter of length of arc of the curve, different for the various integrals. If the function $g(s)$ be assumed to present no singularities, the approximate value of the integral (2.1) can be found by utilizing the method of stationary phase as applied by Rubinowicz [3] to the scalar case. This value is

$$\int_{\Gamma} e^{-ik\zeta} g(s)\, ds = e^{ik\zeta_\nu} g(s_\nu) \sqrt{\frac{2\pi}{k\zeta_\nu''}}\, e^{-i\frac{\pi}{4}}. \tag{2.2}$$

The index ν is employed for denoting that the value of a function is taken at the point $s = s_\nu$, where $\zeta'(s) = 0$.

It is easily proved that

$$\zeta'' = \sin^2(\varrho, ds)\left(\frac{1}{r}+\frac{1}{\varrho}\right) + \frac{1}{K}\left[\cos(r, K)+\cos(\varrho, K)\right], \tag{2.3}$$

with K denoting the radius of curvature directed along the principal normal to Γ.

Now, this method ceases to be applicable to the integral in the neighbourhood of the boundary of shadow, since the function $g(s)$ presents here a singularity (indeed, at the shadow boundary, $r\varrho+\vec{r}\vec{\varrho} = 0$).

This difficulty, however, can be avoided [3], and the following formula is obtained for the integral of (1.2) near the shadow boundary within the cone of light:

$$u_B^{(1)} = -\frac{e^{i\left(kR+\frac{3}{4}\pi\right)}}{R\sqrt{2}} \int_{-\infty}^{\sqrt{\frac{2k}{\pi}(\rho+r-R)}} e^{-i\frac{\pi}{2}v^2}\, dv. \tag{2.4}$$

The integral of (1.2) near the geometrical shadow boundary and in the shadow is given by

$$u_B^{(2)} = \frac{e^{-i\left(kR+\frac{3}{4}\pi\right)}}{R\sqrt{2}} \int_{\infty}^{\sqrt{\frac{2k}{\pi}(\rho+r-R)}} e^{-i\frac{\pi}{2}v^2}\, dv. \tag{2.5}$$

Moreover, the total light wave within the light cone can be expressed as follows:

$$u = u_B^{(1)} + \frac{e^{-ikR}}{R} = \frac{e^{-i\left(kR+\frac{3}{4}\pi\right)}}{R\sqrt{2}} \int_{\infty}^{-\sqrt{\frac{2k}{\pi}(\rho+r-R)}} e^{-i\frac{\pi}{2}v^2}\, dv. \tag{2.6}$$

Approximate formulas for the diffracted dipole wawe

In the present section, the foregoing approximative method will be applied to obtain approximate values of the vectors $\vec{E}$ and $\vec{H}$ defined by Eqs. (1.1). The discussion will be given for the x-component of $\vec{E}$. By (1.1),

$$E_x(P) = E_x^*(P) + \frac{1}{4\pi}\frac{\partial}{\partial x_L}\frac{\partial}{\partial z_L} u^B(L,P) + \frac{1}{4\pi}\int_{\dot{\Gamma}} (\vec{ds}\times\vec{E}_0)_x \frac{e^{-ikr}}{r} + \frac{1}{4\pi ik}\frac{\partial}{\partial x_P}\int_{\dot{\Gamma}} (\vec{ds}\,\vec{H}_0)\frac{e^{-ikr}}{r}, \tag{3.1}$$

where

$$\vec{E}_0 = \frac{1}{4\pi}\operatorname{grad}_Q \frac{\partial}{\partial z_Q}\frac{e^{-ik\rho}}{\varrho} + \vec{\varkappa}\frac{k^2}{4\pi}\frac{e^{-ik\rho}}{\varrho},$$

$$\vec{H}_0 = \frac{-ik}{4\pi}\left[\operatorname{grad}_L \frac{e^{-ik\rho}}{\varrho}\times\vec{\varkappa}\right]. \tag{3.2}$$

In the formulas for $\vec{E}_0$ and $\vec{H}_0$, the operations indicated are now carried out, and only the terms proportional to the highest powers of k are retained. All other terms are neglected as small. Easy computation yields

$$\begin{cases} \vec{E}_0 = -\dfrac{k^2}{4\pi}\dfrac{e^{-ik\rho}}{\varrho}\dfrac{\partial\varrho}{\partial z_L}\operatorname{grad}_L\varrho + \vec{\varkappa}\dfrac{k^2}{4\pi}\dfrac{e^{-ik\rho}}{\varrho}, \\ \vec{H}_0 = -\dfrac{k^2}{4\pi}\dfrac{e^{-ik\rho}}{\varrho}[\operatorname{grad}_L\varrho\times\vec{\varkappa}]. \end{cases} \tag{3.2a}$$

We now take up Eq. (3.1) once more. The approximate value of the function $u^B(L,P)$ within the region considered (near the shadow boundary) is known as given by Eqs. (2.4) and (2.5). Thus, we now consider the integral

$$\int_{\dot{\Gamma}} (\vec{ds}\times\vec{E}_0)_x\frac{e^{-ikr}}{r} = \int_{\Gamma} t_y E_{0z}\frac{e^{-ikr}}{r}ds - \int_{\Gamma} t_z E_{0y}\frac{e^{-ikr}}{r}ds.$$

The two right hand integrals can be tackled by applying Eq. (2.2), wherein the part of the function $g(s)$ is played by expressions computed from (3.2) or (3.2a), respectively. Computation yields

$$\int_{\dot{\Gamma}} (\vec{ds}\times\vec{E}_0)_x\frac{e^{-ikr}}{r} = \frac{e^{-i\left(kr+\frac{\pi}{4}\right)}}{r}\sqrt{\frac{2\pi}{k\zeta''}}\,[\vec{t}\times\vec{E}_0]_x\Big|_v, \tag{3.3}$$

wherein $\vec{E}_0$ is given by (3.2) or (3.2a).

The fourth right hand term in Eq. (3.1) is handled likewise. Introducing the operator $\partial/\partial x_p$ under the integral and applying formula (2.2), neglecting the term in $1/r$ as compared with ik, we obtain:

$$\frac{\partial}{\partial x_p}\int_{\dot{\Gamma}} (\vec{ds}\,\vec{H}_0)\frac{e^{-ikr}}{r} = -ik\frac{e^{-i\left(kr+\frac{\pi}{4}\right)}}{r}\frac{\partial r}{\partial x_p}\sqrt{\frac{2\pi}{k\zeta''}}\,(\vec{t}\,\vec{H}_0)\Big|_0 \tag{3.4}$$

In the foregoing formula, $\vec{H}_0$ is given by (3.2) or (3.2a).

Taking into account (2.4), (2.5), (3.3) and (3.4), the following approximate value is obtained for the component E_x given by Eq. (3.1):

$$E_x = E_x^* \pm \frac{e^{-i\frac{3}{4}\pi}}{\sqrt{2}} \frac{\partial}{\partial x_L} \frac{\partial}{\partial z_L} \left(\frac{e^{-ikR}}{R} \int_{\infty}^{\sqrt{\frac{2k}{\pi}(r+\rho-R)}} e^{-i\frac{\pi}{2}v^2}\, dv \right)_v + \frac{1}{4\pi} \frac{e^{-i\left(kr+\frac{\pi}{4}\right)}}{r} \sqrt{\frac{2\pi}{k\zeta''}} [\vec{t} \times \vec{E}_0]_x \Big|_v - \frac{1}{4\pi} \frac{e^{-i\left(kr+\frac{\pi}{4}\right)}}{r} \sqrt{\frac{2\pi}{k\zeta''}} \frac{\partial r}{\partial x_p} (\vec{t}\, \vec{H}_0) \Big|_v .$$

It will be remembered that the term E_x^* (and, similarly, H_x^* for the vector $\vec{H}$) enters the formulas only when points within the light cone are considered; then, too, the sign "—" should be taken before the expression $\frac{\partial}{\partial x_L} \frac{\partial}{\partial z_L} (\ldots\ldots\ldots)_v$. For the region of shadow, this expression is taken with the sign "+".

Since the remaining components of the vector $\vec{E}$ and those of $\vec{H}$ are computed quite similarly, we shall omit them here, restricting ourselves to the final results.

Eqs. (3.1) now assume the approximate form

$$\left\{ \begin{aligned} \vec{E} &= \vec{E}_x^* \pm \mathrm{grad}_L \frac{\partial}{\partial z_L} \left(\frac{e^{-i\left(kR+\frac{3}{4}\pi\right)}}{R\sqrt{2}} \int_{\infty}^{\sqrt{\frac{2k}{\pi}(r+\rho-R)}} e^{-i\frac{\pi}{2}v^2}\, dv \right)_v \pm \\ &\quad - \frac{1}{4\pi} \frac{e^{-i\left(kr+\frac{\pi}{4}\right)}}{r} \sqrt{\frac{2\pi}{k\zeta''}} [\vec{t} \times \vec{E}_0] \Big|_v + \\ &\quad + \frac{1}{4\pi} \frac{e^{-i\left(kr+\frac{\pi}{4}\right)}}{r} \sqrt{\frac{2\pi}{k\zeta''}} (\vec{t}\, \vec{H}_0)\, \mathrm{grad}_p\, r \Big|_v , \\ \vec{H} &= \vec{H}^* \pm k \left[\left[\mathrm{grad}_L \left(\frac{e^{-i\left(kR+\frac{5}{4}\pi\right)}}{R\sqrt{2}} \int_{\infty}^{\sqrt{\frac{2k}{\pi}(r+\rho-R)}} e^{-i\frac{\pi}{2}v^2}\, dv \right)_v \times \vec{\varkappa} \right] + \\ &\quad - \frac{1}{4\pi} \frac{e^{-i\left(kr+\frac{\pi}{4}\right)}}{r} \sqrt{\frac{2\pi}{k\zeta''}} [\vec{t} \times \vec{H}_0] \Big|_v + \\ &\quad - \frac{1}{4\pi} \frac{e^{-i\left(kr+\frac{\pi}{4}\right)}}{r} \sqrt{\frac{2\pi}{k\zeta''}} (\vec{t}\, \vec{E}_0)\, \mathrm{grad}_p\, r \Big|_v . \end{aligned} \right. \tag{3.5}$$

The expressions acted on by the differential operators $\mathrm{grad}_L\, \partial/\partial z$ or grad_L have been placed within brackets bearing the lower index v. This is to indicate that the value of the respective expression in point P_v should be taken first, and the operation of differentiation carried out thereafter.

There is the following remark to be made concerning Eqs. (3.5). Indeed, at first sight, the thought might arise of handling Eqs. (1.1) in the same manner as Rubinowicz [3] handled the scalar case, i.e. by making use of the circumstance that, at the boundary of light and shadow, the following relationship holds:

$$E_x^{(1)} + E_x^* = E_x^{(2)},$$

where $E_x^{(1)}$ and $E_x^{(2)}$ denote the x-components of the vector $\vec{E}$ in the light and shadow, respectively. On setting forth from this point of view, all further procedure would be quite analogous to that of Rubinowicz, with the sole difference that, in the present case, expressions more complicated and integrals more involved would occur because of the second order differentiation in E_x. This procedure, however, is found to lead nowhere, as the integrals it yields are divergent near the boundary of light and shadow and the electromagnetic field assumes infinite values.

Eqs. (3.5) with the terms in $\vec{E}^*$ and $\vec{H}^*$ omitted can be applied throughout the region remote from the boundary of geometrical shadow, as all the terms present therein have explicite finite values. However, on comparing the results obtained in the present case with the strict solution due to Senior [7] describing diffraction of a dipole wave on the edge of an ideally conducting semi-plane, the results of either theory are found to disagree. The same conclusion is reached on comparing the results from Kottler's theory with the well-known strict solution of Sommerfeld describing diffraction of a plane electromagnetic wave on a semi-plane. Such facts should not give rise to astonishment, as long as the very essential differences separating the definitions of the diffracting screen in either theory are kept in mind. However, as will be shown in Part II of the present investigation, these differences vanish if the field near the boundary of geometrical shadow is considered.

The region remote from the boundary of geometrical shadow is of lesser interest from the standpoint of physics, since diffraction phenomena occur preponderantly near the boundary of light and shadow.

Part II of the present paper will bring a discussion of Eqs. (3.5) for this region.

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THEORETICAL PHYSICS

Green Functions of the Mixed Cauchy Problem of the Gordon-Klein Equation and Their Connection with Advanced and Retarded Green Functions

by

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The Green functions of the Gordon-Klein equation are very well known and there exists a vast literature concerning this problem. But the basic Green functions namely advanced and retarded, have been so far usually introduced as particular solutions of the Gordon-Klein equation in a fashion which could be named "Deus ex machina". The purpose of this paper is to show that these functions are intimately connected with the Green function of the mixed Cauchy problem. The presented method of construction of the above mentioned functions has some interesting features.

Let us consider the following problem: the Gordon-Klein equation

$$(\Box - \varkappa^2)\, u(x_k, x_0) = 4\pi\, \varrho\, (x_k, x_0) \tag{1}$$

is to be solved in the whole space and the function $u(x_k, x_0)$ should satisfy the following additional conditions: the boundary condition

$$u(x_k, x_0) \to 0 \quad \text{for} \quad (x_l\, x_l)^{1/2} \to \infty, \tag{2}$$

the initial-value condition

$$u(x_k, \tilde{x}_0) = \tilde{u}_0, \qquad \left.\frac{\partial u(x_k, x_0)}{\partial x_0}\right|_{x_0 = \tilde{x}_0} = \frac{\partial \tilde{u}_0}{\partial x_0} \tag{3}$$

(the mixed Cauchy problem).

Making use of the linearity of the Gordon-Klein equation and of the additional conditions we may divide our problem into two parts.

In the first part the homogeneous Gordon-Klein equation is to be solved with nonhomogeneous condition of the mixed Cauchy problem. This part is very easy to calculate and its solution is well known. It may be written in the form

$$u_1 = \int d_{(3)}\, x' \left[\Delta\,(x_l - x'_l, x_0 - \tilde{x}_0)\, \frac{\partial \tilde{u}_0}{\partial x_0} + \frac{\partial \Delta}{\partial x_0}\, \tilde{u}_0\,(x'_l, \tilde{x}_0) \right], \tag{4}$$

where $\Delta(x_k, x_0)$ is often called the Green function of the homogeneous Gordon-Klein equation and has the following integral form:

$$\Delta(x_k, x_0) = \frac{1}{(2\pi)^3} \int d_{(3)} k e^{ik_l x_l} \frac{\sin \sqrt{k^2+\varkappa^2}\, x^0}{\sqrt{k^2+\varkappa^2}}. \tag{5}$$

In the second part let us consider the solution of nonhomogeneous Gordon-Klein equation with homogeneous additional condition of the mixed Cauchy problem. This will be done by means of the Green function of the Gordon-Klein equation satisfying the same homogeneous additional conditions. This function must be a solution of the following equation

$$(\Box - \varkappa^2)\, G(x_k, x_0 \,|\, y_k, y_0) = \delta_{(3)}(x_k - y_k)\, \delta(x_0 - y_0) \tag{6}$$

and will be sought in the form of Fourier integral:

$$G(x_k, x_0/y_k, y_0) = \frac{1}{(2\pi)^4} \int e^{ik_l(x_l - y_l) - ik_0(x_0 - y_0)} g(k_l, k_0)\, d_{(4)} k. \tag{7}$$

Inserting Eq. (7) into Eq. (6) we obtain an equation for $g(k_l, k_0)$

$$(-k_l k_l + k_0^2 - \varkappa^2)\, g(k_l, k_0) = 1, \tag{8}$$

where we have used the usual Fourier integral for δ function

The general solution of this equation has the form [1]:

$$g(k_l, k_0) = P \frac{1}{k_0^2 - k_l k_l - \varkappa^2} + \lambda(k_l, k_0)\, \delta(k_0^2 - k_l k_l - \varkappa^2), \tag{9}$$

where the function $\lambda(k_l, k_0)$ should be determined from the additional conditions. Boundary condition is satisfied automatically. Thus, let us write the initial-value conditions:

$$-P \int \frac{e^{ik_l(x_l - y_l) - ik_0(\tilde{x}_0 - y_0)}}{k_0^2 - k_l k_l - \varkappa^2}\, d_{(4)} k = \int e^{ik_l(x_l - y_l) - ik_0(\tilde{x}_0 - y_0)} \lambda(k_l, k_0)\, \delta(k_0^2 - k_l k_l - \varkappa^2)\, d_{(4)} k, \tag{10}$$

$$-P \int \frac{e^{ik_l(x_l - y_l) - ik_0(\tilde{x}_0 - y_0)}(-ik_0)}{k_0^2 - k_l k_l - \varkappa^2}\, d_{(4)} k = \int e^{ik_l(x_l - y_l) - ik_0(\tilde{x}_0 - y_0)} (-ik_0)\, \lambda(k_l, k_0)\, \delta(k - k_l k_l - \varkappa^2)\, d_{(4)} k. \tag{11}$$

When we use the fact that

$$\delta(k_0^2 - k_l k_l - \varkappa^2) = \frac{\delta(k_0 - \sqrt{k_l k_l + \varkappa^2}) + \delta(k_0 + \sqrt{k_l k_l + \varkappa^2})}{2|k_0|}$$

and perform the integration with respect to k_0. Eqs. (10) and (11) take the form:

$$(10') \quad \pi \frac{\tilde{x}_0 - y_0}{|x_0 - y_0|} \int e^{ik_l(x_l - y_l)} \frac{\sin\sqrt{k^2+\varkappa^2}(\tilde{x}_0 - y_0)}{\sqrt{k^2+\varkappa^2}} d_{(3)}k =$$

$$= \frac{1}{2} \int e^{ik_l(x_l - y_l)} \left[\frac{e^{-i\sqrt{k^2+\varkappa^2}(\tilde{x}_0 - y_0)}\lambda(k_l, +)}{\sqrt{k^2+\varkappa^2}} + \frac{e^{i\sqrt{k^2+\varkappa^2}(\tilde{x}_0 - y_0)}\lambda(k_l, -)}{\sqrt{k^2+\varkappa^2}} \right] d_{(3)}k,$$

$$(11') \quad \pi \frac{\tilde{x}_0 - y_0}{|\tilde{x}_0 - y_0|} \int e^{ik_l(x_l - y_l)} \cos\sqrt{k^2+\varkappa^2}(\tilde{x}_0 - y_0) d_{(3)}k =$$

$$= -\frac{i}{2} \int e^{ik_l(x_l - y_l)} [e^{-i\sqrt{k^2+\varkappa^2}(\tilde{x}_0 - y_0)}\lambda(k_l, +) - e^{i\sqrt{k^2+\varkappa^2}(\tilde{x}_0 - y_0)}\lambda(k_l, -)] d_{(3)}k,$$

where

$$\lambda(k_l, \pm) \equiv \lambda(k_l, \pm\sqrt{k^2+\varkappa^2}).$$

Since exponential functions form a complete set of orthogonal functions, Eqs. (10′) and (11′) are equivalent to the following equations:

$$(12) \quad \pi \frac{\tilde{x}_0 - y_0}{|x_0 - y_0|} \sin\sqrt{k^2+\varkappa^2}(\tilde{x}_0 - y_0) = \frac{1}{2} [e^{-i\sqrt{k^2+\varkappa^2}(\tilde{x}_0 - y_0)}\lambda(k_l, +) + e^{i\sqrt{k^2+\varkappa^2}(\tilde{x}_0 - y_0)}\lambda(k_l, -)],$$

$$(13) \quad \pi \frac{\tilde{x}_0 - y_0}{|x_0 - y_0|} \cos\sqrt{k^2+\varkappa^2}(\tilde{x}_0 - y_0) = -\frac{i}{2} [e^{-i\sqrt{k^2+\varkappa^2}(\tilde{x}_0 - y_0)}\lambda(k_l, +) - e^{i\sqrt{k^2+\varkappa^2}(\tilde{x}_0 - y_0)}\lambda(k_l, -)],$$

wherefrom, after simple calculations, we obtain:

$$(14) \quad \lambda(k_l, \mp) = \mp i\pi\,\epsilon(\tilde{x}_0 - y_0), \quad \text{where} \quad \epsilon(x_0 - y_0) \equiv \frac{\tilde{x}_0 - y_0}{|x_0 - y_0|}.$$

Now we may calculate the ultimate form of the Green function of the Gordon-Klein equation satisfying the homogeneous conditions of the mixed Cauchy problem:

$$(15) \quad G(x_k, x_0 | y_k, y_0) = \frac{P}{(2\pi)^4} \int \frac{e^{ik_l(x_l - y_l) - ik_0(x_0 - y_0)}}{k_0^2 - k_l k_l - \varkappa^2} d_{(4)}k +$$

$$+ \frac{1}{(2\pi)^4} \int \frac{e^{ik_l(x_l - y_l)}}{2\sqrt{k^2+\varkappa^2}} [e^{-i\sqrt{k^2+\varkappa^2}(x_0 - y_0)}\lambda(k_l, +) + e^{i\sqrt{k^2+\varkappa^2}(x_0 - y_0)}\lambda(k_l, -)] d_{(3)}k =$$

$$= \frac{1}{(2\pi)^4}(-\pi)\frac{x_0 - y_0}{|x_0 - y_0|} \int e^{ik_l(x_l - y_l)} \frac{\sin\sqrt{k^2+\varkappa^2}(x_0 - y_0)}{\sqrt{k^2+\varkappa^2}} d_{(3)}k +$$

$$+ \frac{i}{(2\pi)^4} \frac{\pi}{2} \frac{\tilde{x}_0 - y_0}{|\tilde{x}_0 - y_0|} \int \frac{e^{ik_l(x_l - y_l)}}{\sqrt{k^2+\varkappa^2}} [e^{-i\sqrt{k^2+\varkappa^2}(x_0 - y_0)} - e^{i\sqrt{k^2+\varkappa^2}(x_0 - y_0)}] d_{(3)}k =$$

$$= -\frac{\pi}{(2\pi)^4} \left[\frac{x_0 - y_0}{|x_0 - y_0|} - \frac{\tilde{x}_0 - y_0}{|\tilde{x}_0 - y_0|} \right] \int e^{ik_l(x_l - y_l)} \frac{\sin\sqrt{k^2+\varkappa^2}(x_0 - y_0)}{\sqrt{k^2+\varkappa^2}} d_{(3)}k =$$

$$= -\frac{1}{2} \left[\frac{x_0 - y_0}{|x_0 - y_0|} - \frac{\tilde{x}_0 - y_0}{|\tilde{x}_0 - y_0|} \right] \Delta(x_k - y_k, x_0 - y_0).$$

The full expression for the solution of the mixed Cauchy problem of the Gordon-Klein equation has the following form:

$$u(x_k, x_0) = -2\pi \int d_{(4)} y \left[\frac{x_0 - y_0}{|x_0 - y_0|} - \frac{\tilde{x}_0 - y_0}{|\tilde{x}_0 - y_0|} \right] \Delta(x_k - y_k, x_0 - y_0) \times \times \varrho(y_k, y_0) + \int d_{(3)} y \left[\Delta(x_k - y_k, x_0 - \tilde{x}_0) \frac{\partial \tilde{u}_0}{\partial x_0} + \frac{\partial \Delta}{\partial x_0} \tilde{u}(y_k, \tilde{x}_0) \right]. \tag{16}$$

Let us consider now the mixed Cauchy problem in which the time x_0 is determined as $-\infty$ and the initial values $\tilde{u}_0$ and $\partial \tilde{u}_0 / \partial x_0$ vanish (this means for example that the particle which generates the field $u(x_k, x_0)$ is in spatial infinity at the time $\tilde{x}_0 = -\infty$). In this case the expression (16) consists only of the first term. The Green function which is involved in this term is the solution of the second part of our preceding problem.

Let us pass to the limit $\tilde{x}_0 \to -\infty$. Our Green function takes now the following form:

$$G_R(x_k, x_0 | y_k, y_0) = \begin{cases} -\Delta(x_k - y_k, x_0 - y_0) & x_0 > y_0 \\ 0 & x_0 < y_0. \end{cases}$$

This is exactly the retarded Green function. The derivation is consistent with the physical meaning of the problem.

It is also very interesting to pass with $\tilde{x}_0$ to $+\infty$ with the same assumptions concerning $\tilde{u}_0$ and $\partial \tilde{u}_0 / \partial x_0$ as before. Our Green function now has the following form

$$G_A(x_k, x_0 | y_k, y_0) = \begin{cases} 0 & x_0 > y_0 \\ \Delta(x_k - y_k, x_0 - y_0) & x_0 < y_0. \end{cases}$$

This is the advanced Green function whose derivation is again consistent with the physical meaning of the problem.

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EXPERIMENTAL PHYSICS

Yield of Fluorescence and Spectra of Chlorophyll in Viscous Media

by

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The emission and absorption spectra of chlorophyll in various media have been investigated by a number of authors [1]—[5]. However, data on the fluorescence of chlorophyll in transparent viscous solvents at room temperature are scarce ([1] p. 732, [4], [6]).

The present paper deals with the emission and absorption spectra and relative yield of fluorescence at various exciting wavelengths of viscous solutions of chlorophyll *a*. Chlorophyll *a* was extracted from dried nettle leaves and purified chromatographically in the sugar column [7]. The degree of purity of the chlorophyll thus obtained was checked spectrophotometrically by comparing its absorption spectrum with that given in the literature [8].

The viscous medium consisted of collodium. A number of samples of different viscosity were prepared. Figs. 1, 3 and and 4 show the results of measurements for solutions of a concentration of 1.2×10^{-5} g/cm^3 and a viscosity of 3.8 and 31 poise.

The intensity distribution of the fluorescence excited with a high-pressure HBO 200 mercury lamp was investigated with a double Müller-Hilger monochromator and an RCA 1P photomultiplier of known spectral sensitivity.

The fluorescence spectra of the solutions investigated are shown in Fig. 1 (in relative units). The height of the maximum investigated for chlorophyll in ether was 2.9 times greater than for chlorophyll in collodium. In Fig. 1, the fluorescence spectrum of chlorophyll in ether has been plotted in reduced scale.

In the experimental arrangement and at the concentrations used the error from reabsorption and secondary fluorescence did not appreciably affect the shape of the graphs.

The relative yield of fluorescence of chlorophyll *a* in collodium as referred to that of the same dye in ether was measured by a method described in an earlier paper [9].

The device is shown in Fig. 2. The concentration of the solution used for comparison was chosen for each exciting wavelength so that it should exhibit the same

absorption as the solution under investigation, when measured with the photomultiplier (in position A of Fig. 2). To obtain equal absorption, the solution used for comparison had to be of a concentration about one half of that of the solution investigated (4.5×10^{-6} g/cm^3).

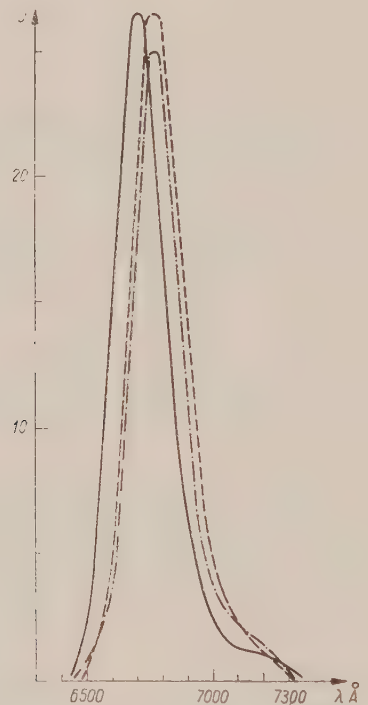

Fig. 1. Fluorescence spectra of chlorophyll *a*.

———————— in ether solution

— — — — — in collodium of 3.8 poise viscosity

—·—·—·— in collodium of 31 poise viscosity

The measured relative yields are shown in Fig. 3. The width of the spectral interval of the exciting light amounted to approximately 60 Å. The error of the yield measurements arising from differences in the degree of polarization of fluorescence between either solution did not exceed 5%. Using the absolute fluorescence

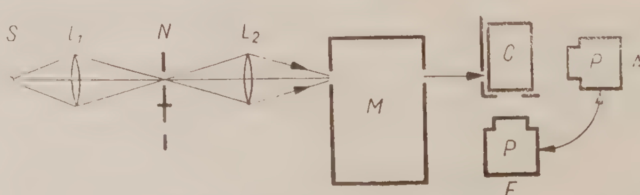

Fig. 2. Setting of device for determining absorption and fluorescence yield. S — 1000 W bulb; L_1, L_2 — lenses; N — light interruptor; M — monochromator; C — container with aperture; P — photomultiplier; A — setting of photomultiplier for absorption measurement; F — setting of photomultiplier for fluorescence intensity measurement

yields of chlorophyll *a* in ether given by L. Forster and R. Livingston [10], the relative yields measured at Stokes excitation (straight line section of graphs in Fig. 3) were converted to absolute yields. The results are given in Table I.

TABLE I

Medium	Fluorescence yield at $\lambda_{exc.} = 6440$ Å	
	fresh solution	same solution subsequent to 5 days storage
Collodium of 3.8 poise viscosity	0.145	0.065
Collodium of 31 poise viscosity	0.16	0.084
Ether according to [1], p. 731	0.24	—

The absorption spectra were measured with the same device (Fig. 2). The absorption in the red region of the spectrum of chlorophyll *a* in ether and in collodium is shown in Fig. 4.

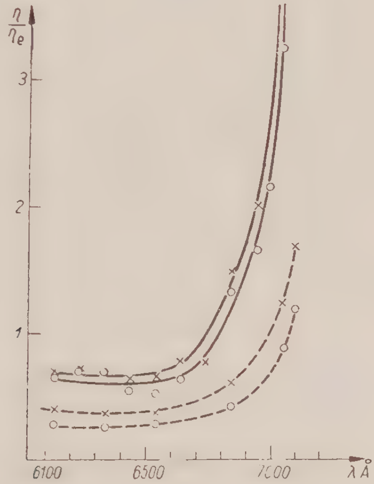

Fig. 3. Relative fluorescence yield of chlorophyll *a* in collodium and in ether, for different exciting wavelengths

⊙ — solution of viscosity of 3.8 poise

× — solution of viscosity of 31 poise

The continuous graphs correspond to freshly prepared solutions; the dashed graphs — to the same solutions stored for 5 days

From Figs. 1, 3 and 4, the graphs corresponding to various viscosities of the collodium are seen not to differ essentially. The principal emission maximum (Fig. 1) in the collodium solutions of either viscosity is at 6715 Å and is shifted by about 50 Å towards longer wavelengths as compared with the maximum of the ether solution.

The fluorescence yield of chlorophyll *a* in collodium (Table I) is lesser than that in ether, and decreases as the time interval separating preparation of the solution and measurement increases (Table I). For the solution of higher viscosity, the rate of this decrease is lesser. The absorption in the red absorption band was found also to decrease with time. The continuous graphs in Fig. 3 are for freshly prepared solutions, whilst the dashed ones are for the same solutions subsequent to five days storage at —10°C. The shape of the absolute yield of chlorophyll *a* in collodium was evaluated by extrapolating Forster's data [10] to longer exciting wavelengths. In the anti-Stokes region, the yield decreases.

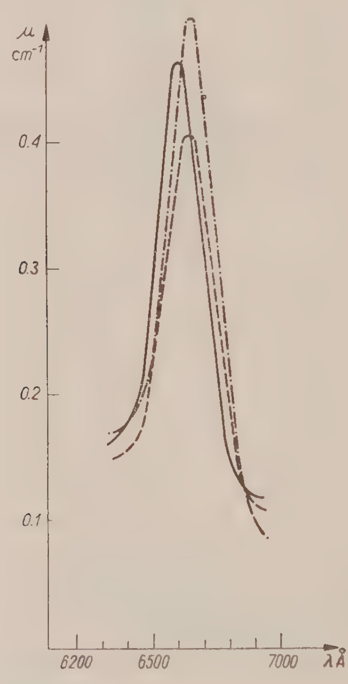

Fig. 4. Absorption spectra of chlorophyll *a*.

——— solution in ether, of concentration $c = 4.5\times 10^{-6}$ g/cm³
— — — solution in collodium of viscosity of 3.8 poise, $c = 1.2\times 10^{-5}$ g/cm³
·· · ·— solution in collodium of viscosity 31 poise, $c = 1.2\times 10^{-5}$ g/cm³

The absorption maximum (Fig. 4) of both solutions of chlorophyll *a* in collodium lies at 6665 Å, and, similarly to the fluorescence maximum, is shifted by some 50 Å towards longer wavelengths with respect to the maximum in eher solution.

The fact that all the graphs for either collodium solution resemble one another suggests that the differences between the graphs for the ether solution and those for the collodium solutions are due to the presence of collodium molecules rather than to differences in viscosity. Even practically rigid solutions in collodium, when investigated with Becquerel's phosphoroscope, exhibited no luminescence that could be ascribed to phosphorescence or to slow fluorescence. This is in accordance

with the Becker and Fernandez hypothesis [11] that, in the case of chlorophyll, the band due to a transition from the metastable to the ground level is observed in solutions containing no hydroxyl group.

As aggregates of chlorophyll molecules are formed, the absorption and emission of the solution diminishes (see, e.g. Yefremova's results [3]). A chlorophyll *a* solution in collodium revealed lesser absorption within the red band and weaker emission than a chlorophyll solution in ether that had been prepared in an identical way. Hence, presumably the degree of aggregation of the chlorophyll molecules is greater in collodium than in ether. By Foerster [12] and Jabłoński's [13] hypothesis, dimers of the dye account for the fall in yield in the anti-Stokes region. As the concentration of non-fluorescent dimers increases, the drop in yield becomes more marked. However, no conclusions as to association of the chlorophyll molecules can be drawn from the shape of the yield graph, since the exact shape of the yield curve at anti-Stokes excitation in an ether solution of chlorophyll *a* is not known as yet.

The shift in the maxima of the emission and absorption bands (amounting to about 50 Å) is of the order of the shifts in the chlorophyll spectra due to a change of solvent (cf. e.g. [8] and [14]).

Results obtained point to some kind of aggregation of the chlorophyll molecules in collodium solution, (of, [3] and [1], p. 714). The viscosity of the solution is not essential in determining the course of the phenomenon, and only slows down the process of aggregation (cf. the graphs in Fig. 3). A final interpretation will be proposed after similar measurements for other viscous and rigid solvents are carried out.

Note added in proof: The scale of wevelengths in Figs. 1, 3 and 4 is to be shifted by 50 Å towards longer wavelengths.

The authors wish to thank Professor Dr A. Jabłoński for his kind interest and advice throughout the present investigation.

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EXPERIMENTAL PHYSICS

Electroluminescence of ZnSe-Cu with a Cu_2Se Layer in an A.C. and D.C. Field

by

H. ŁOŻYKOWSKI

Presented by A. JABŁOŃSKI on July 18, 1960

Introduction

According to papers [1]—[3] non-electroluminescent phosphors ZnS-Cu, CdS-Ag and ZnSe-Cu when coated with a thin semiconductor layer of higher conductivity become electroluminescent. O. Kazankin et al. [4] observed electroluminescence of ZnS-Cu, Mn containing Cu_2S in the form of a distinct phase, in an a.c. and d.c. field.

The present paper describes certain properties of the ZnSe-Cu electroluminophor coated with a thin layer of Cu_2Se.

Experimental

A ZnSe-Cu cathodoluminophor (1 g ZnSe — 10^{-3} g Cu) was washed in a copper nitrate solution and dried.

The copper nitrate bath led to formation of a Cu_2Se layer on the ZnSe-Cu.

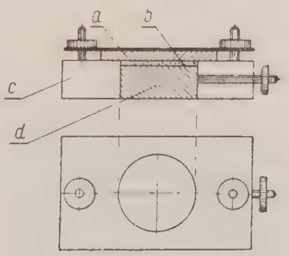

Fig. 1. Diagram of electroluminescent cell. *a* — semiconducting glass; *b* — layer of luminophor in dielectric; *c* — plexiglass; *d* — copper electrode

The electroluminescence of the phosphor thus prepared was investigated on placing the latter within an electroluminescent cell (Fig. 1) in castor oil or paraffin suspension. The luminophor layer thickness amounted to 0.35 mm in oil, and to 0.20 mm in paraffin.

The electroluminescence was investigated at a sinusoidal voltage ranging from 0 to 400 V and 100 Hz to 15 kHz, and at a pulsating voltage and d.c. voltage utilizing a 1P 21 photomultiplier Fig. 2 shows block diagram of the device.

Results

A. On applying the a. c. voltage. the investigated luminophor exhibited red luminescence. On applying a d. c. field, the luminescence was visually observed to be shifted towards shorter wavelengths.

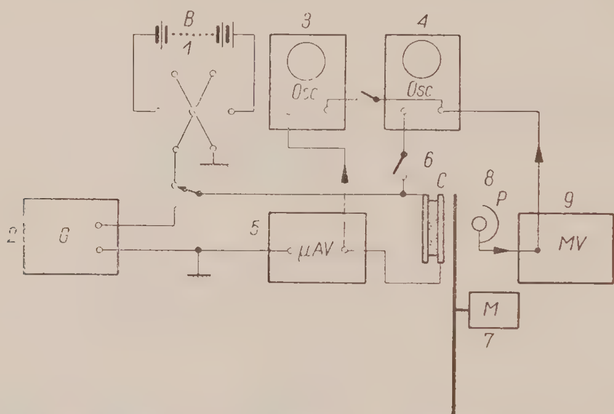

Fig. 2. Block diagram of the device. 1 — anode battery; 2 — generator; 3,4 — double beam oscillographs; 5 — valve microammeter; 6 — electroluminescent cell; 7 — mechanical interruptor: 8 — photomultiplier; 9 — valve millivoltmeter

Fig. 3 contains the results of measurements of intensity of electroluminescence B *versus* the voltage applied for three different frequencies (Fig. 3 a) and for the d. c. voltage (Fig. 3 b), the phosphor being suspended in castor oil. The ordinates represent the natural logarithm of the intensity of luminescence, whilst the abscissae bring the values of $10/\sqrt{V}$.

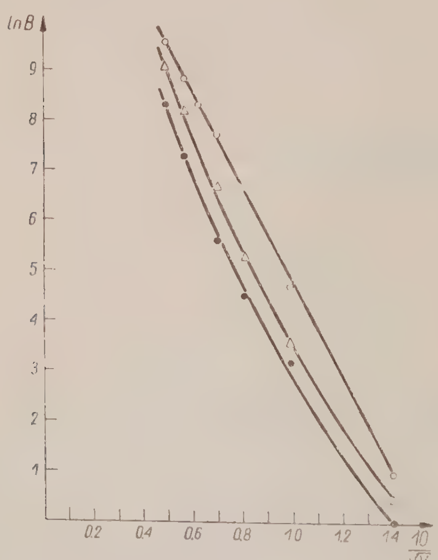

Fig. 3a. Voltage dependence of electroluminescence. ○ 500 Hz, ● 5 KkHz, △ 20 kHz

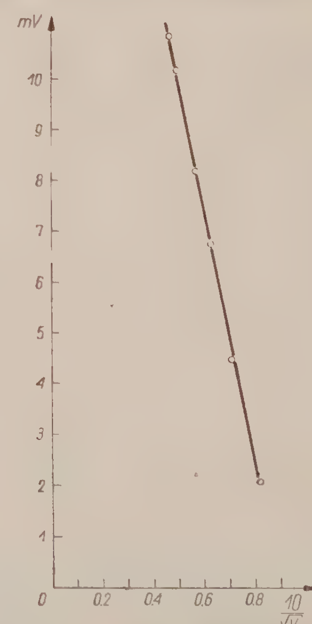

Fig. 3b. Voltage dependence of electroluminescence, at d. c. voltage

The graph in Fig. 3a shows that the relation

$$B = Ae^{b\sqrt{V}}$$

is not satisfied for all frequencies. Simultaneously with the intensity of luminescence, the current flowing through the electroluminescent cell was measured as *versus* the strength of the applied electric field.

The results for ln I *versus* $\sqrt{E}$ are given in Fig. 4.

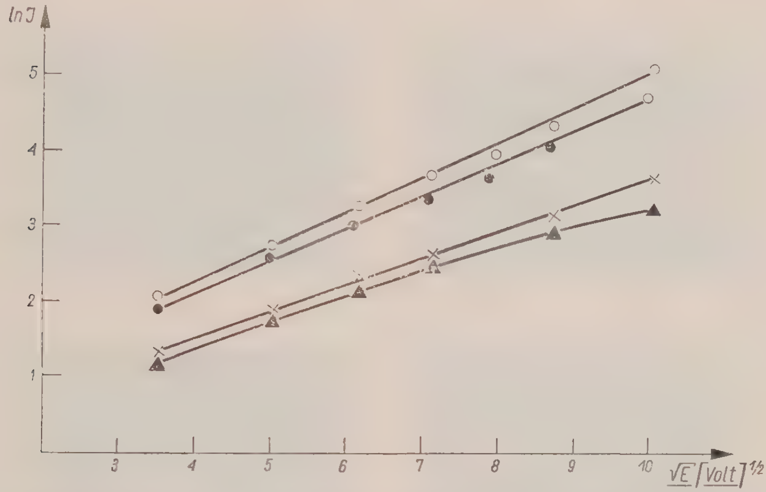

Fig. 4. Ln I $\sqrt{E}$ graph for current through cell. ○ 500 Hz, × 5 kHz, ▲ 20 kHz, ● d. c. voltage

The oscillogram of the current flowing through the cell at a frequency of 500 Hz is shown in Fig. 5.

Frequency dependence of the brightness waves is shown in Fig. 6 (*a*, *b*, *c*, *d*, *e*).

From the oscillograms, the second maximum is seen to increase with the frequency, and to exceed the principal maximum at 4 kHz.

The photos in Fig. 6 *f*, *g* illustrate brightness waves at 3 kHz and 400 V.

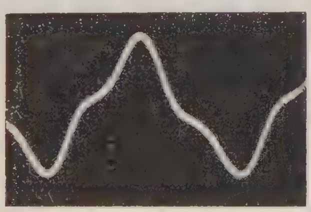

Fig. 5. Oscillogram of current flowing through cell at 200 V and 500 Hz

Fig. 6f shows brightness waves observed with an interference filter presenting a transmission maximum at 5013 Å and a width of 106 Å, whilst Fig. 6 *g* was obtained with no filter.

B. As mentioned above, the electroluminescence of ZnSe-Cu coated with a Cu_2Se layer was also investigated in paraffin suspension. The electroluminescent

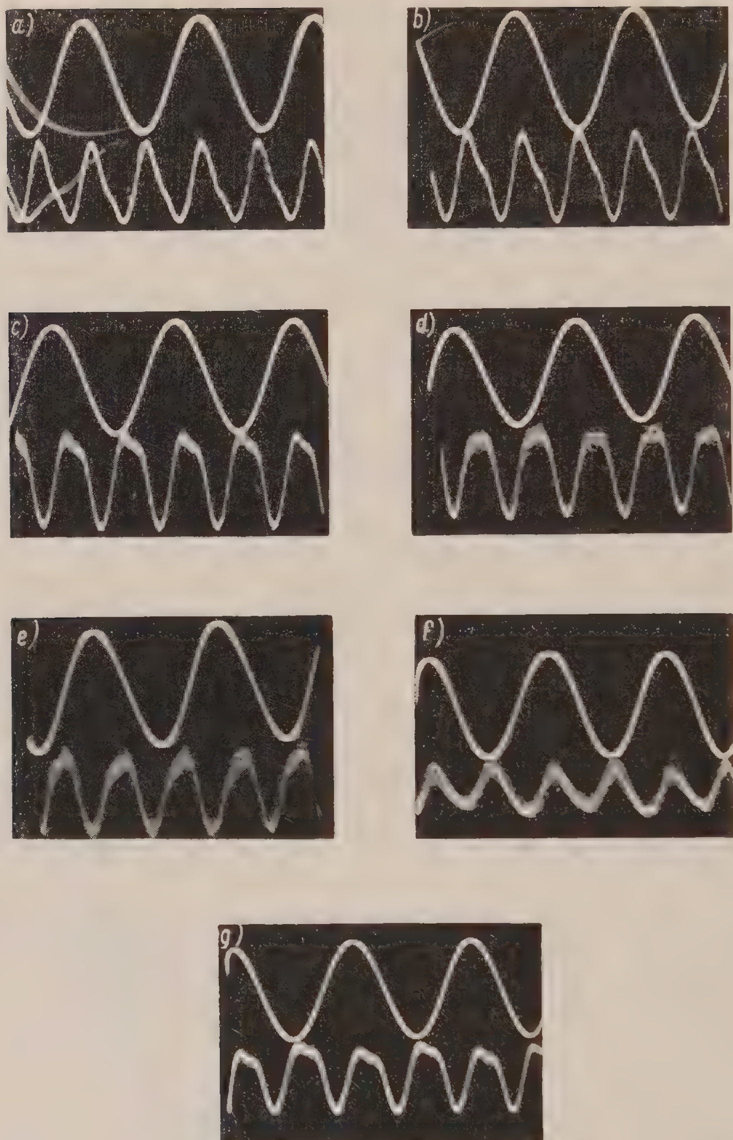

Fig. 6. Brightness waves at 300 V (*a*, *b*, *c*, *d*) *versus* the frequency. *a* — 1 kHz, *b* — 2 kHz, *c* — 3 kHz, *d* — 4 kHz, *e* — 5 kHz, *f* — 3 kHz, 400 V with interference filter, *g* — 3 kHz, 400 V without interference filter

cell with paraffin as a dielectric was prepared as follows: the cell with 0.2 mm cavity Fig. 1, filled with the electroluminophor, was placed in molten paraffin and was covered after degassing of the phosphor, with semiconduoting glass. The cell was then taken out of the molten paraffin and a d. c. voltage of 150 V applied to the electrodes until the paraffin solidified [5].

During and after the process of solidification the cell emitted yellow electroluminescence.

The cell obtained by the foregoing method possesses the properties of a current rectifier (Fig. 7).

When the voltage is applied in the forward direction of conduction a greater luminous intensity is observed than when it is applied in the reverse direction. If the current exceeds a threshold value, the intensity of luminescence is proportional to the current (Fig. 8) flowing through the cell, in agreement with observations carried out on SiC [6], ZnS [7], CdS [8], GaP [9], and other compounds.

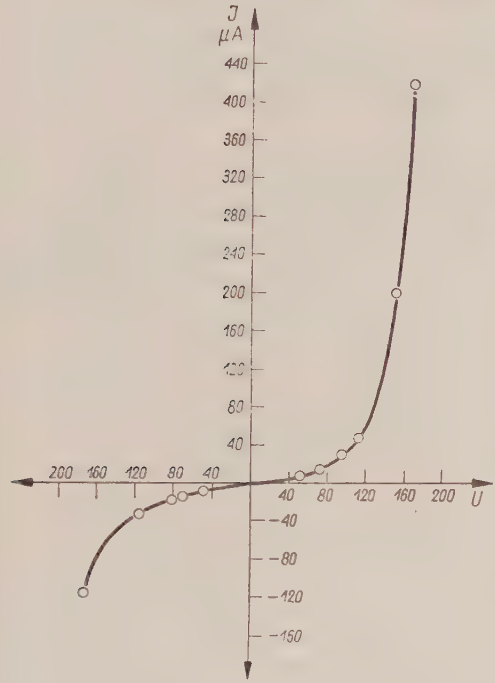

Fig. 7. Current-voltage dependence for an electroluminescent cell with paraffin dielectric.

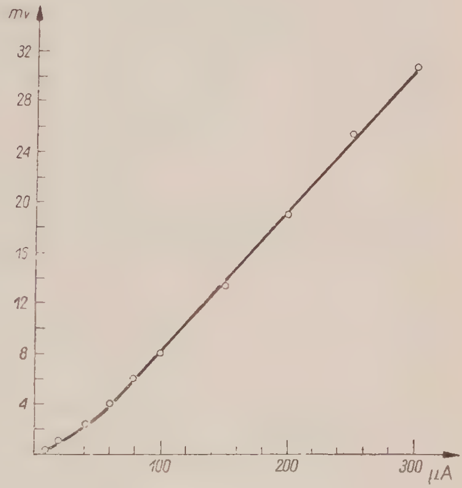

Fig. 8. Luminous intensity *versus* current flowing through cell.

Discussion of results

Electroluminescence of Cu_2Se-coated ZnSe-Cu may be observed from fields of $1.4 \times 10^{+3}$ V/cm upwards.

In powdered phosphors luminescence is observed to occur near the surface, where the local field considerably exceeds the mean field strength value. Strong local fields are related to the presence of a distinct phase of higher conductivity on the phosphor grain surface (Cu_2S, Cu_2Se).

In the case of the paraffin dielectric cell, the Cu_2Se-coated ZnSe-Cu exhibits, in addition to electroluminescence proper, an electroluminescent component related to the injection of electrons into the strong field region (collision mechanism of excitation [7]).

This is confirmed by the following linear dependence of the emitted intensity of luminescence on the current flowing through the cell

$$B = Ia - b.$$

As has already been said, the colour of the luminescence emitted in such cells is shifted towards shorter wavelengths: this phenomenon may be related to the recombination of electrons with the activator centre, whose energy levels lie nearer the valency band [8]*).

The form of the brightness waves observed in Cu_2Se-coated ZnSe-Cu in castor oil is the same whether the cell contains an isolating layer (mica) or not.

The observed increase of the second maximum with the frequency is related to the presence of the Cu_2Se layer on the ZnSe-Cu crystals, as ZnSe-Cu without a Cu_2Se layer showed no such dependence.

As may be seen from the graph in Fig. 4, the field strength dependence of the current flowing through the cell containing the phosphor in the form of a suspension in castor oil coincides satisfactory with the expression proposed by Frenkel for thermo-electric ionisation. According to Frenkel, ionisation energy decreases on applying the field and is given by the expression

$$\omega_i - \Delta\omega = \omega_i - 2e\sqrt{\frac{eE}{\varepsilon_\infty}},$$

where ω_i denotes ionisation energy, and ε_∞ — dielectric constant.

The probability of thermal ionisation

$$p = \nu \exp\left(\frac{-\omega_i - 2e\sqrt{\frac{eE}{\varepsilon_\infty}}}{kT}\right)$$

increases with E.

The author wishes to thank Professor A. Jabłoński for his valuable discussions and advice in connection with the present investigation.

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*) An exact spectral investigation is in preparation

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EXPERIMENTAL PHYSICS

Electroluminescence of Alkaline Earth Sulphide Phosphors

by

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Presented by A. JABŁOŃSKI on July 18, 1960

Results obtained by A. Wachtel [1], who produced electroluminescence in certain alcaline earth sulphides, were utilized in an attempt to produce electroluminophors on a BaS and SrS basis.

The phenomenon of electroluminescence in phosphors, considered until recently to be non-electroluminescent, is related to the presence of a semiconductor layer such as Cu_2S or Ag_2S in the form of a separate phase.

Such a layer can be formed by one of the following methods:

1. by washing the photoluminophor in a solution of copper halide in anhydrous methyl alcohol, or
2. by heating the phosphor with excess activator in an atmosphere of H_2S.

Electroluminophors obtained by the former method exhibit no electroluminescence when the semiconducting layer [1] is washed away.

In order to produce BaS and SrS electroluminophors, the latter method was adopted.

BaS and SrS electroluminophors were obtained by heating barium and strontium carbonate with an admixture of sulphur, flux and activator, in an atmosphere of hydrogen sulphide.

The substrates were ground in a mortar, alcohol being added to obtain a homogeneous distribution of the activator, and then dried at about 100°C.

On drying, the phosphor was placed in a silica tube and heated at 1050°C for half an hour in a stream of H_2S.

The following phosphors were obtained:

1. BaS-Cu with an admixture of KBr flux, exhibiting yellow electroluminescence. Samples of various Cu concentration were prepared. The best electroluminescence was obtained with a concentration of 1% mole Cu per mole of BaS.
2. BaS-Cu (1% mole/mole) — Sm (0.03% mole/mole) with an admixture of KBr flux, exhibiting yellow electroluminescence of a stronger glow than the former.

3. BaS-Sm (0.03% mole/mole) with an admixture of KBr flux, exhibiting orange electroluminescence.
4. SrS-Ag (0.64% mole/mole) with Na_2SO_4 flux, exhibiting blue-green electroluminescence.

All electroluminophors obtained exhibited electroluminescence in fields of greater strength than is the case with ZnS or CdS phosphors.

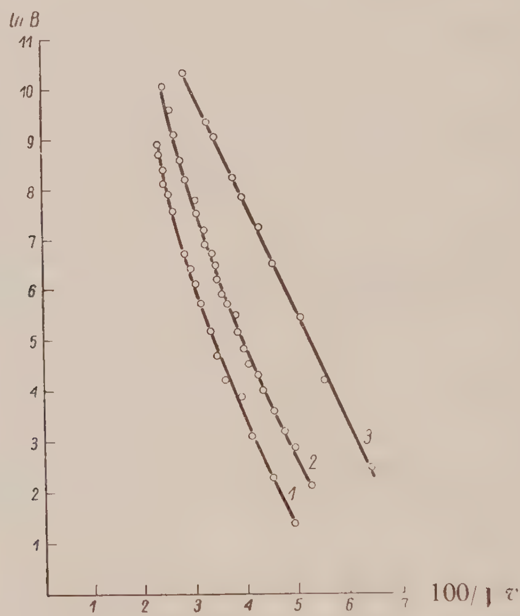

Fig. 1a. Brightness of electroluminescence *versus* voltage applied, for BaS-Cu(1%) phosphor. Graph 1 — BaS-Cu (1%) at frequency of 500 Hz; graph 2 — BaS-Cu (1%) at 5 kHz; graph 3 — BaS-Cu (1%) — Sm (0.03%) at 1 kHz.

The voltage dependence of the electroluminescence of the BaS-Cu, BaS-Cu-Sm and BaS-Sm phosphors is shown in Figs. 1a and 1b. The natural logarithms of luminous intensity are plotted on the ordinates and the values of $100/\sqrt{V}$ on the abscissae.

Experimental results are in good agreement with the following expression for the voltage dependence of the luminous intensity [2]:

$$B = A \cdot e^{-b/\sqrt{V}}.$$

Measurements of the luminous intensity dependence of the frequency of the electric field applied were also carried out for the produced electroluminophors, at one and the same voltage of 1000 V, by the following method.

The signal from a 1P 21 photomultiplier was fed to an oscillograph, and the pulse obtained was measured on the oscillograph screen. In order to determine both the constant and variable components, a mechanical interruptor was introduced between the photomultiplier and the electroluminescent cell.

Results for the BaS-Cu electroluminophor are shown in Fig. 2.

From the graphs, the constant component (curve c) is seen to increase with the frequency, whereas the variable component (curve b) first increases attaining saturation, whilst the total intensity of electroluminescence increases almost linearly throughout the investigated range of frequencies.

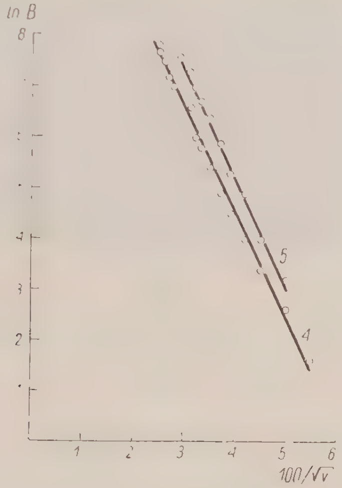

Fig. 1b. Brightness of electroluminescence *versus* voltage for BaS-Sm (0.03%) phosphor. Graph 1 — BaS-Sm (0.03%) at 800 Hz; graph 2 — BaS-Sm (0.03%) at 8 kHz.

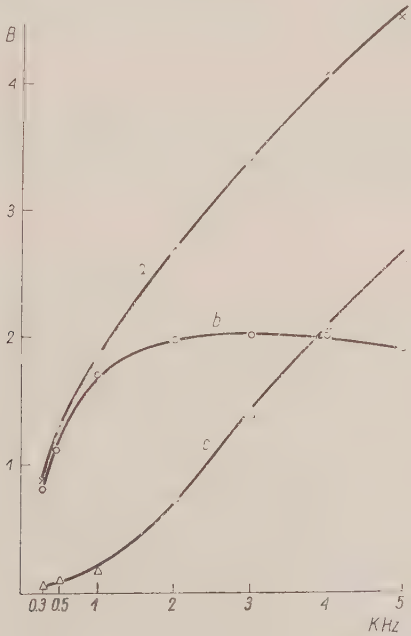

Fig. 2. Brightness of electroluminescence *versus* frequency. Graph *a* — total luminous intensity of BaS-Cu (1%) electroluminophor, at 1000 V; graph *b* — variable component; graph *c* — constant component.

The oscillogram of Fig. 3 illustrates brightness waves, typical of the BaS-Cu electroluminophor.

Research work undertaken for the purpose of finding the activator leading to the highest yield in electroluminophors on a BaS and SrS basis, and on the effect of the flux on the brightness of electroluminescence, is under way.

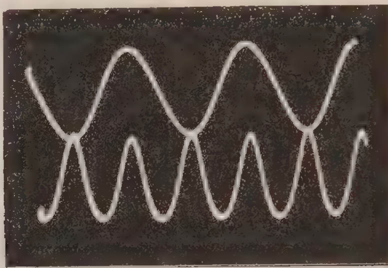

Fig. 3. Oscillogram of brightness wave, for BaS-Cu (1%) phosphor, at 1000 V and 1 kHz.

The authors wish to thank Professor A. Jabłoński for his discussions and helpful suggestions throughout the present investigation.

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БЮЛЛЕТЕНЬ ПОЛЬСКОЙ АКАДЕМИИ НАУК

СЕРИЯ МАТЕМАТИЧЕСКИХ, АСТРОНОМИЧЕСКИХ И ФИЗИЧЕСКИХ НАУК

1960

ТОМ VIII — Резюме статей — ВЫПУСК 10

Ч. ОЛЕХ, **ЗАМЕЧАНИЯ ОБ УСЛОВИЯХ ЕДИНСТВЕННОСТИ РЕШЕНИЙ ОБЫКНОВЕННЫХ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ** . . стр. 661—666

Рассмотрим дифференциальное уравнение

(1) $y' = f(x, y)$ и первоначальное условие $y(0) = 0$,

где x и y — действительные переменные, а $f(x, y)$ является непрерывной функцией в множестве D; $0 \leqslant x \leqslant a$, $|y| \leqslant b$.

Следующее неравенство

$$|f(x, y_1) - f(x, y_2)| \leqslant h(x, |y_1 - y_2|) \tag{2}$$

дает условие единственности для решений уравнения (1), если функция $h(x, u)$ выполняет одно из следующих предположений:

Условие Перрина. $h(x, u)$ непрерывна для $0 \leqslant x \leqslant a$, $u \geqslant 0$, причем уравнение

$$u' = h(x, u) \tag{3}$$

обладает единственным решением $u(x) \equiv 0$ выходящим из точки $(0, 0)$.

Условие Камке. $h(x, u)$ непрерывна в множестве $0 < x < a$, $u \geqslant 0$ и функция $u(x) \equiv 0$ является единственной непрерывной функцией для $0 \leqslant x < \gamma$, выполняющей (3) в интервале $(0, \gamma)$, для произвольного $\gamma > 0$, для которой существует $D_+ u(0)$ и $D_+ u(0) = u(0) = 0$.

Очевидно, что условие Камке более общее, чем условие Перрина.

Автор показывает, что если непрерывная функция $f(x, y)$ выполняет неравенство (2) с функцией $h(x, u)$, выполняющей условие Камке, то существует функция $h_f(x, u)$, выполняющая условие Перрона, для которой также имеется неравенство аналогичное неравенству (2). Таким образом, автор показывает, что условие Камке является в некотором смысле бесполезно слишком общим.

Ч. ОЛЕХ, **О СУЩЕСТВОВАНИИ И ЕДИНСТВЕННОСТИ РЕШЕНИЙ ОБЫКНОВЕННОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ В СЛУЧАЕ ПРОСТРАНСТВА БАНАХА** . . . . стр. 667—673

Пусть $h(x, u) \geqslant 0$ — непрерывная функция по отношению к u при каждом фиксированном x и измеримая по отношению к x при каждом фиксированном u; кроме того, пусть для каждого множества ограниченного S, содержащегося в множестве $V: 0 < x \leqslant a$, $0 \leqslant u$ существует функция $\chi_S(x)$ измеримая по Лебегу в интервале $\langle 0, a \rangle$ и суммируемая на каждом интервале $\langle \gamma, a \rangle$ $(\gamma > 0)$ и такая, что $h(x,u) \leqslant \chi_S(x)$ для $(x, u) \epsilon S$. Наконец, предположим, что единственной абсолютно непрерывной функцией $u(x)$, равной нулю для $x = 0$, для которой существует $D_+ u(0)$ и равняется нулю и выполняющей почти везде в интервале $(0, \gamma)$ для произвольного $0 < \gamma < a$ уравнение $u' = h(x, u)$ является функция идентично равная нулю.

Рассмотрим дифференциальное уравнение

$$y' = f(x, y), \tag{1}$$

где x является действительным, y принадлежит к линейному и полному пространству Банаха, а $f(x, y)$ является непрерывной и ограниченной функцией в множестве $W: x_0 \leqslant x \leqslant x_0 + a$, $|y - y_0| \leqslant b$ принимающей значения из пространства Банаха.

При предположении

$$|f(x, y_1) - f(x, y_2)| \leqslant h(x, |y_1 - y_2|)$$

в множестве W автор показывает существование и единственность решения уравнения (1), выходящего из точки (x_0, y_0). Результаты, полученные автором представляют собой обобщение недавно опубликованных результатов, полученных Т. Важевским.

ЧИ ГУАН-ФУ, **ЗАМЕТКА О СИНГУЛЯРНЫХ ИНТЕГРАЛАХ ВЕКТОРНЫХ ФУНКЦИЙ** . . . . стр. 675—679

В работе рассматривается сходимость в среднем сингулярных интегралов векторных функций при норме Орлича. Некоторые леммы Орлича ([6], стр. 129—132) обобщены для случая векторных функций, пренебрегая одновременно Δ_2- уловием.

Классическая теорема М. Рисса о средней сходимости ряда Фурье не может быть распространена на векторные функции, тем не менее аналоги теорем Джексона и Кведа для векторных функций — справедливы.

Я. МИКУСИНСКИЙ, **ЗАМЕТКА О ЗНАЧЕНИИ ДИСТРИБУЦИИ В ТОЧКЕ**

стр. 681—683

В работе [3] автором было дано определение нерегулярных операций на функциях. Каждая нерегулярная операция может быть распространена в операцию на дистрибуциях. Значение функции в некотсрой точке оказывается нерегулярной операцией. Следовательно, путем ее распространения, получается понятие значения дистрибуции.

В настоящей работе доказывается, что полученное распространение равносильно определению значения, данному С. Лоясевичем [1], [2].

Р. СИКОРСКИЙ, **ДЕТЕРМИНАНТНАЯ ТЕОРИЯ ТИПА КАРЛЕМАНА**

стр. 685—689

Развита теория детерминантной системы, т.е. системы состоящей из определителя и миноров всех порядков для оператора $I+T$, действующего в гильбертовом пространстве, где I — тождественный оператор и T — оператор порожденный матрицей $(\tau_{i,j})$, обладающей свойством $\sum_{i,j} \tau_{i\,j}^{2} < \infty$, если рассматривать гильбертово пространство как пространство l^2. Введение определителя совпадает с определителями, изученными Карлеманом в случае интегральных уравнений в пространстве L^2 и Смитесом в случае абстрактного гильбертова пространства. Изучена связь введенных миноров с минорами Карлемана. Приведены формулы на решение уравнения $x + Tx = x_0$ и сопряженного с ним уравнения, даже в случае, когда определитель равен нулю.

Р. СИКОРСКИЙ, **О ПОДСТАНОВКАХ В δ-ФУНКЦИИ ДИРАКА**

стр. 691—694

Пусть $\delta_q(x)$ — δ-функция Дирака в q-мерном пространстве, $x=(\xi_1, \ldots, \xi_q)$. Пусть $\sigma(x)=(\sigma_1(x), \ldots, \sigma_p(x))$ — отображение открытого подмножества q-мерного пространства в p-мерное пространство, удовлетворяющее условиям: 1° $p < q$, 2° функции $\sigma_j(x)$ $(j=1,2,\ldots,p)$ — бесконечно дифференцируемы и 3° функция

$$J(x)=\sqrt{\sum_{j_1,\ldots,j_p}\left(\frac{\partial(\sigma_1,\ldots,\sigma_p)}{\partial(\xi_{j_1},\ldots,\xi_{j_p})}\right)^2}$$

нигде не обращается в нуль. Тогда имеет место равенство

$$\delta_p(\sigma(x))=\int_S J(t)^{-1}\,\delta_q(x-t)\,dt,$$

где S — $(q-p)$-мерная поверхность $S=\{x: a(x)=0\}$.

Интеграл стоящий в правой части равенства является частным случаем интеграла из обобщенных функций, определенного автором в работе „Integrals of distributions", которая будет опубликована в Studia Mathematica.

Я. С. ЛИПИНЬСКИЙ, **ОБ ОДНОЙ ПРОБЛЕМЕ Э. МАРЧЕВСКОГО, КАСАЮЩЕЙСЯ ПЕРИОДИЧЕСКИХ ФУНКЦИЙ** . . . . стр. 695—697

Э. Марчевский поставил следующий вопрос: существует ли такая последовательность действительных чисел $\{a_n\}$, что для любой ограниченной последовательности чисел $\{b_n\}$ существует непрерывная периодическая функция, принимающая в точках a_n значения b_n.

В работе дан положительный ответ на этот вопрос. А именно доказана следующая теорема:

Пусть $\delta_n > 0$ $(n = 1, 2, ...)$ и $\sum_{n=1}^{\infty} \delta_n = C < \infty$. Если последовательность $\{a_n\}$ удовлетворяет условию

$$\frac{a_{n+1}}{a_n} \geqslant \frac{C + \delta_{n+2}}{\delta_{n+1}} \qquad (n = 1, 2, ...),$$

то для любой ограниченной последовательности $\{b_n\}$ существует локально монотонная непрерывная периодическая функция $f(x)$ такая, что $f(a_n) = b_n \quad (n = 1, 2, ...)$.

А. ЯСЬКЕВИЧ и Г. КОНВЕНТ, **ФЕРРО- И АНТИФЕРРОЭЛЕКТРИЧЕСКИЕ СИСТЕМЫ В ВЕЩЕСТВАХ ТИПА ПЕРОВСКИТА** . . . . стр. 699—702

Представлен метод исследования систем диполей ферроэлектрических кристаллов типа ABO_3. Исследованы системы диполей, которые образуются при переходе из кубической в тетрагональную фазу в случае, когда термический диполь образовался при помощи иона B.

Показано, что для кристалла с переносным ионом B, при переходе из кубической фазы в тетрагональную, возможны три антиферроэлектрические системы и одна — ферроэлектрическая. Приводятся условия реализации отдельных систем.

Б. КАРЧЕВСКИЙ, **ПРИБЛИЖЕННЫЕ ФОРМУЛЫ ДЛЯ ДИФФРАКЦИОННОЙ ЭЛЕКТРО-МАГНИТНОЙ ВОЛНЫ. I.** . . . . стр. 703—708

К известным формулам Коттлера, определяющим диффракцию дипольной электро-магнитной волны, применялся метод стационарной фазы и получены приближенные формулы для значений векторов E и H. Полученные результаты являются правильными для широкого класса диффракционных экранов и вид их значительно более простой, чем вид первоначальных формул Коттлера.

Продискутированы также полученные результаты с точки зрения известных нам точных решений диффракционных проблем.

Р. ЖЕЛЯЗНЫ, **ФУНКЦИЯ ГРИНА СМЕШАННОЙ ПРОБЛЕМЫ КОШИ ДЛЯ УРАВНЕНИЯ ГОРДОНА-КЛЕЙНА И ЕЕ СВЯЗЬ С ЗАПАЗДЫВАЮЩЕЙ И ОПЕРЕЖАЮЩЕЙ ФУНКЦИЕЙ ГРИНА** . . . . стр. 709--712

В работе выводится, при применении интегрального преобразования Фурье, явный вид функции Грина смешанной проблемы Коши для уравнения Гордона-Клейна.

Эта функция удовлетворяет однородным начальным условиям в произвольном моменте $\tilde{x}_0$ и выражается известной функцией Δ уравнения Гордона-Клейна. Асимптотический переход $\tilde{x}_0 \to \pm\infty$ дает запаздывающую и опережающую функцию Грина для вышеупомянутого уравнения.

Д. ФРОНЦКОВЯК и Т. МАРШАЛЭК, **ВЫХОД ФЛУОРЕСЦЕНЦИИ И СПЕКТРЫ ХЛОРОФИЛЛА В ВЯЗКИХ СРЕДАХ** . . . . стр. 713--717

Были измерены эмиссионные и абсорбционные спектры, а также зависимость относительного выхода флуоресценции от длины волны возбуждающего света для раствора хлорофилла в коллодии при двух разных вязкостях.

Полученные результаты объясняются различной ассоциацией молекул хлорофилла в исследованных растворах.

Г. ЛОЖИКОВСКИЙ, **ЭЛЕКТРОЛЮМИНЕСЦЕНЦИЯ СОЕДИНЕНИЯ** ZnSe-Cu **ПОКРЫТОГО СЛОЕМ** Cu_2Se **В ПОСТОЯННОМ И ПЕРЕМЕННОМ ЭЛЕКТРИЧЕСКОМ ПОЛЕ**. . . . . стр. 719--724

В работе дается отчет об исследованиях, проведенных над электролюминесценцией катодного люминофора ZnSe-Cu, покрытого прослойкой Cu_2Se в виде эмульсии в касторовом масле, а также в парафине, причем применялось постоянное и переменное напряжение поля.

Электролюминофор ZnSe-Cu покрытый Cu_2Se помещался в особой коробке в жидком парафине; к элеоктродам приводилось напряжение, которое удерживали до момента застывания парафина.

Образованный таким способом электролюминесцентный элемент обнаруживает выпрямляющие свойства. Интенсивность свечения пропорциональна силе тока, проплывающего через элемент.

Г. ЛОЖИКОВСКИЙ и Г. МЕНЧИНСКАЯ, **ЭЛЕКТРОЛЮМИНЕСЦЕНЦИЯ ЩЕЛОЧНЫХ ФОСФОРНЫХ СУЛЬФАТОВ** . . . . . . стр. 725—728

В работе дается отчет об экспериментах, в результате которых были получены следующие электролюминофоры BaS-Cu, Bas-Sm, Bas-Cu-Sm, SrS-Ag.

Для перечисленных соединений была исследована зависимость их электролюминесценции от напряжения и частоты электрического поля.

TABLE DES MATIÈRES

Cena zł 20.—